PDE I – Problem Set 1. Distributed Wed 9/3/2014, due Tues 9/16/2014.

(1) Recall from Lecture 1 that for a 1D random walk with spatial step Δx, time step Δt, and probability 1/2 of going left or right, the evolution of the probability density is a finite-difference discretization of $u_t = u_{xx}$ when $\frac{(\Delta x)^2}{2\Delta t} = 1$.

(a) Consider the biased random walk in which a walker at $j\Delta x$ moves to $(j+1)\Delta x$ with probability $\frac{1}{2} + \alpha\Delta x$ and moves to $(j-1)\Delta x$ with probability $\frac{1}{2} - \alpha\Delta x$? Assuming as before that $\frac{(\Delta x)^2}{2\Delta t} = 1$, and taking α to be constant, what PDE does the probability density solve in the continuum limit $\Delta x \to 0$?

(b) Now suppose the bias is position-dependent; in other words, using the notation of part (a), suppose $\alpha = \alpha(j\Delta x)$ is a smooth but non-constant function of position. Extend what you found in part (a) to this case. [Warning: note that when $\alpha(x)$ is not constant, $\alpha u_x \neq (\alpha u)_x$.]

(2) In the Lecture 1 notes, the discussion of convection and diffusion ("motivation 1") is quite different from the discussion of probability ("motivation 2"), and there is no discussion about how population dynamics leads to a reaction-diffusion equation. This two-part question provides some amplification.

(a) Use an argument similar to that of "motivation 1" to show that if the population density is $u(x,t)$, the birth rate is $f_1(u)$, the death rate is $f_2(u)$, and diffusion is governed by Fick's law, then $u_t = D\Delta u + f(u)$ with $f = f_1 - f_2$.

(b) Reconcile the apparently different treatments of "motivation 1" and "motivation 2" by showing that if $u_j^n = u(j\Delta x, n\Delta t)$ satisfies

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} = \frac{(\Delta x)^2}{2\Delta t} \, \frac{u_{j+1}^n + u_{j-1}^n - 2u_j^n}{(\Delta x)^2}$$

then for any $j < k$ and any n,

$$\Big[u(j\Delta x, t) + u((j+1)\Delta x, t) + \cdots + u(k\Delta x, t)\Big]_{t=n\Delta t}^{t=(n+1)\Delta t} = \frac{u_{k+1}^n - u_k^n}{2} - \frac{u_j^n - u_{j-1}^n}{2}.$$

Why is this analogous to the conservation law considered in "motivation 1"?

(3) The Lecture 1 notes use an "energy-type" argument to show that if Ω is a bounded domain in R^n, then the evolution problem

$$u_t - \Delta u = f \text{ in } \Omega \text{ for } t > 0, \text{ with } u = \phi \text{ at } \partial\Omega \text{ and } u = u_0 \text{ at } t = 0$$

has at most one solution. They also show that if f and ϕ are independent of time, and accepting the existence of a steady-state solution $\overline{u}$ (solving $-\Delta\overline{u} = f$ in Ω, with $\overline{u} = \phi$ at $\partial\Omega$), we have

$$\frac{d}{dt}\int_\Omega |u - \overline{u}|^2 \, dx \leq -C \int_\Omega |u - \overline{u}|^2 \, dx \qquad (1)$$

with $C > 0$, so that $u - \overline{u} \to 0$ exponentially fast (in L^2) as $t \to 0$. Let's examine what happens when the boundary condition is of Neumann rather than Dirichlet type.

(a) Use an "energy-type" argument to show that if Ω is a bounded domain in R^n, then the evolution problem

$$u_t - \Delta u = f \text{ in } \Omega \text{ for } t > 0, \text{ with } \tfrac{\partial u}{\partial n} = \psi \text{ at } \partial\Omega \text{ and } u = u_0 \text{ at } t = 0 \qquad (2)$$

has at most one solution. In the special case $f = 1$, $u_0 = 0$, $\psi = 0$, can you write down the solution explicitly?

(b) Now suppose f and ψ are independent of time. Observe that for the steady-state problem ($-\Delta\overline{u} = f$ in Ω with $\frac{\partial \overline{u}}{\partial n} = \psi$ at $\partial\Omega$) to have a solution, f and ψ must satisfy the *consistency condition* $\int_{\partial\Omega} \psi\, ds + \int_\Omega f\, dx = 0$. Also that if the steady-state problem has a solution at all, then it is non-unique since for any constant c, $\overline{u} + c$ is another solution. Assuming the existence of a steady-state solution (something we'll prove later on; we are assuming here of course that f and ψ are consistent), show that when u solves (2) it satisfies an estimate of the form (1), with $\overline{u}$ chosen so that $\int_\Omega \overline{u}\, dx = \int_\Omega u_0\, dx$. [Note: you'll need to use the mean-value-zero analogue of Poincaré's inequality. It asserts the existence of a constant M_Ω such that $\int_\Omega g^2\, dx \le M_\Omega \int_\Omega |\nabla g|^2\, dx$ for any function g such that $\int_\Omega g\, dx = 0$. You may use this result without proving it. Incidentially: the best choice of M_Ω is $1/\lambda$, where λ is the first nonzero eigenvalue of the Laplacian with a Neumann boundary condition at $\partial\Omega$.]

(c) Finally, suppose f and ψ are independent of time but inconsistent in the sense that $\int_{\partial\Omega} \psi\, ds + \int_\Omega f\, dx \ne 0$. What is the large-time behavior of the solution of (2)?

(4) In the Lecture 1 notes, "motivation 4" was the modeling of heat transfer. In that setting, a physically natural assumption is that the heat flux at the boundary is proportional to the difference between the temperature $u(x,t)$ and some fixed constant U. Known as "Newton's law of cooling", this models loss of heat by radiation, if the far-field temperature is U. So let's consider the heat equation $u_t = \Delta u$ in a bounded domain $\Omega \subset R^n$, with initial condition $u = u_0(x)$ at $t = 0$ and boundary condition

$$\tfrac{\partial u}{\partial n} = -k(u - U) \text{ at } \partial\Omega.$$

where k and U are constants.

(a) Assuming that $k > 0$, use an "energy-type argument" to show that this evolution problem has at most one solution.

(b) When $k < 0$ the same conclusion is valid, but you'll have to work harder to prove it. Give a proof based on the assertion that for any $\varepsilon > 0$ there exists $C_\varepsilon > 0$ such that

$$\int_{\partial\Omega} v^2\, ds \le C_\varepsilon \int_\Omega v^2\, dx + \varepsilon \int_\Omega |\nabla v|^2\, dx. \qquad (3)$$

(c) Prove the estimate (3) for a domain with a sufficiently smooth boundary. [Hint: one argument begins by choosing a smooth vector field σ on Ω such that $\sigma =$ outward unit normal at $\partial\Omega$. Then $\int_{\partial\Omega} v^2\, ds = \int_{\partial\Omega} v^2 \sigma \cdot n\, ds = \int_\Omega \operatorname{div}(\sigma v^2)\, dx = \cdots$.]

(5) Consider the semilinear equation

$$u_t - \Delta u = u^5,$$

in a bounded domain $\Omega \subset R^n$, with Dirichlet boundary condition $u = 0$ at $\partial\Omega$ and initial condition $u(x,0) = u_0(x)$. Show that if

$$E[u_0] = \int_\Omega \left(\frac{1}{2}|\nabla u_0|^2 - \frac{1}{6}u_0^6 \right) dx < 0$$

then the solution "blows up," i.e. a classical solution ceases to exist in finite time. [Hint: start by noting that $\frac{d}{dt}E[u(t)] \leq 0$. Then derive a relation linking $\frac{d}{dt}\int_\Omega u^2\,dx$ with $E[u(t)]$.]