
(1) Recall from Lecture 1 that for a 1D random walk with spatial step $\Delta x$, time step $\Delta t$, and probability $1/2$ of going left or right, the evolution of the probability density is a finite-difference discretization of $u_t = u_{xx}$ when $\frac{(\Delta x)^2}{2\Delta t} = 1$.

(a) Consider the biased random walk in which a walker at $j\Delta x$ moves to $(j + 1)\Delta x$ with probability $\frac{1}{2} + \alpha \Delta x$ and moves to $(j - 1)\Delta x$ with probability $\frac{1}{2} - \alpha \Delta x$? Assuming as before that $\frac{(\Delta x)^2}{2\Delta t} = 1$, and taking $\alpha$ to be constant, what PDE does the probability density solve in the continuum limit $\Delta x \to 0$?

(b) Now suppose the bias is position-dependent; in other words, using the notation of part (a), suppose $\alpha = \alpha(j \Delta x)$ is a smooth but non-constant function of position. Extend what you found in part (a) to this case. [Warning: note that when $\alpha(x)$ is not constant, $\alpha u_x \neq (\alpha u)_x$.]

(2) In the Lecture 1 notes, the discussion of convection and diffusion (“motivation 1”) is quite different from the discussion of probability (“motivation 2”), and there is no discussion about how population dynamics leads to a reaction-diffusion equation. This two-part question provides some amplification.

(a) Use an argument similar to that of “motivation 1” to show that if the population density is $u(x,t)$, the birth rate is $f_1(u)$, the death rate is $f_2(u)$, and diffusion is governed by Fick’s law, then $u_t = D u_{xx} + f_1(u) - f_2(u)$.

(b) Reconcile the apparently different treatments of “motivation 1” and “motivation 2” by showing that if $u^n_j = u(j \Delta x, n \Delta t)$ satisfies

$$\frac{u^{n+1}_j - u^n_j}{\Delta t} = \frac{(\Delta x)^2}{2\Delta t} \left( \frac{u^n_{j+1} + u^n_{j-1} - 2u^n_j}{(\Delta x)^2} \right)$$

then for any $j < k$ and any $n$,

$$\left[ u(j \Delta x, t) + u((j + 1) \Delta x, t) + \cdots + u(k \Delta x, t) \right]_{t=n\Delta t} = \frac{u^n_{k+1} - u^n_k}{2} - \frac{u^n_k - u^n_{k-1}}{2}.$$

Why is this analogous to the conservation law considered in “motivation 1”?

(3) The Lecture 1 notes use an “energy-type” argument to show that if $\Omega$ is a bounded domain in $\mathbb{R}^n$, then the evolution problem

$$u_t - \Delta u = f \text{ in } \Omega \text{ for } t > 0, \text{ with } u = \phi \text{ at } \partial \Omega \text{ and } u = u_0 \text{ at } t = 0$$

has at most one solution. They also show that if $f$ and $\phi$ are independent of time, and accepting the existence of a steady-state solution $\pi$ (solving $-\Delta \pi = f$ in $\Omega$, with $\pi = \phi$ at $\partial \Omega$), we have

$$\frac{d}{dt} \int_{\Omega} |u - \pi|^2 \, dx \leq -C \int_{\Omega} |u - \pi|^2 \, dx$$

with $C > 0$, so that $u - \pi \to 0$ exponentially fast (in $L^2$) as $t \to 0$. Let’s examine what happens when the boundary condition is of Neumann rather than Dirichlet type.
(a) Use an “energy-type” argument to show that if $\Omega$ is a bounded domain in $\mathbb{R}^n$, then the evolution problem

\[ u_t - \Delta u = f \text{ in } \Omega \text{ for } t > 0, \text{ with } \frac{\partial u}{\partial n} = \psi \text{ at } \partial \Omega \text{ and } u = u_0 \text{ at } t = 0 \quad (2) \]

has at most one solution. In the special case $f = 1, u_0 = 0, \psi = 0$, can you write down the solution explicitly?

(b) Now suppose $f$ and $\psi$ are independent of time. Observe that for the steady-state problem

\[ (-\Delta u = f \text{ in } \Omega \text{ with } \frac{\partial u}{\partial n} = \psi \text{ at } \partial \Omega) \]

to have a solution, $f$ and $\psi$ must satisfy the consistency condition $\int_{\partial \Omega} \psi \, ds + \int_{\Omega} f \, dx = 0$. Also that if the steady-state problem has a solution at all, then it is non-unique since for any constant $c$, $u + c$ is another solution.

Assuming the existence of a steady-state solution (something we’ll prove later on; we are assuming here of course that $f$ and $\psi$ are consistent), show that when $u$ solves (2) it satisfies an estimate of the form (1), with $u$ chosen so that $\int_{\Omega} u \, dx = \int_{\Omega} u_0 \, dx$. [Note: you’ll need to use the mean-value-zero analogue of Poincaré’s inequality. It asserts the existence of a constant $M_\Omega$ such that $\int_{\Omega} g^2 \, dx \leq M_\Omega \int_{\Omega} |\nabla g|^2 \, dx$ for any function $g$ such that $\int_{\Omega} g \, dx = 0$. You may use this result without proving it. Incidentally: the best choice of $M_\Omega$ is $1/\lambda$, where $\lambda$ is the first nonzero eigenvalue of the Laplacian with a Neumann boundary condition at $\partial \Omega$.]

(c) Finally, suppose $f$ and $\psi$ are independent of time but inconsistent in the sense that $\int_{\partial \Omega} \psi \, ds + \int_{\Omega} f \, dx \not= 0$. What is the large-time behavior of the solution of (2)?

(4) In the Lecture 1 notes, “motivation 4” was the modeling of heat transfer. In that setting, a physically natural assumption is that the heat flux at the boundary is proportional to the difference between the temperature $u(x, t)$ and some fixed constant $U$. Known as “Newton’s law of cooling”, this models loss of heat by radiation, if the far-field temperature is $U$. So let’s consider the heat equation $u_t = \Delta u$ in a bounded domain $\Omega \subset \mathbb{R}^n$, with initial condition $u = u_0(x)$ at $t = 0$ and boundary condition

\[ \frac{\partial u}{\partial n} = -k(u - U) \text{ at } \partial \Omega. \]

where $k$ and $U$ are constants.

(a) Assuming that $k > 0$, use an “energy-type argument” to show that this evolution problem has at most one solution.

(b) When $k < 0$ the same conclusion is valid, but you’ll have to work harder to prove it. Give a proof based on the assertion that for any $\varepsilon > 0$ there exists $C_\varepsilon > 0$ such that

\[ \int_{\partial \Omega} v^2 \, ds \leq C_\varepsilon \int_{\Omega} v^2 \, dx + \varepsilon \int_{\Omega} |\nabla v|^2 \, dx. \quad (3) \]

(c) Prove the estimate (3) for a domain with a sufficiently smooth boundary. [Hint: one argument begins by choosing a smooth vector field $\sigma$ on $\Omega$ such that $\sigma = \text{outward unit normal at } \partial \Omega$. Then $\int_{\partial \Omega} v^2 \, ds = \int_{\partial \Omega} v^2 \sigma \cdot n \, ds = \int_{\Omega} \text{div}(\sigma v^2) \, dx = \cdots$.]

(5) Consider the semilinear equation

\[ u_t - \Delta u = u^5, \]

in a bounded domain $\Omega \subset \mathbb{R}^n$, with Dirichlet boundary condition $u = 0$ at $\partial \Omega$ and initial condition $u(x, 0) = u_0(x)$. Show that if

\[ E[u_0] = \int_{\Omega} \left( \frac{1}{2} |\nabla u_0|^2 - \frac{1}{6} u_0^6 \right) \, dx < 0 \]
then the solution “blows up,” i.e. a classical solution ceases to exist in finite time. [Hint: start by noting that $\frac{d}{dt} E[u(t)] \leq 0$. Then derive a relation linking $\frac{d}{dt} \int \Omega u^2 \, dx$ with $E[u(t)]$.]