PDE – Lecture 9, 11/11/2014

Goal for Lectures 9+10: linear wave eqn $u_{tt} - \Delta u = 0$ + related matters.

Essential character of this eqn (quite different from the heat eqn) is its "Hamiltonian" character: eqn is a ple version of "force = mass × accel". Also the (related) fact that "kinetic + potential energy" is conserved, eg

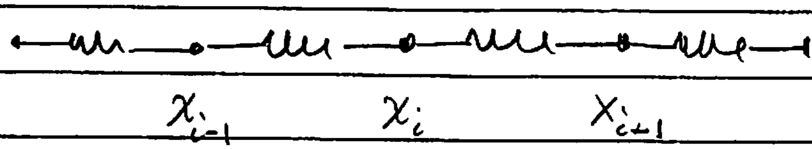
$$\text{if } \quad u_{tt} - \Delta u = 0 \quad \text{for } x \in \Omega$$
$$u = 0 \quad \text{at } \partial\Omega$$

$$\text{then } \quad \frac{d}{dt} \frac{1}{2} \int_\Omega u_t^2 + |\nabla u|^2 \, dx$$
$$= \int_\Omega u_t u_{tt} + \nabla u \cdot \nabla u_t$$
$$= \int_\Omega u_t (u_{tt} - \Delta u) + \cancelto{0}{\int_{\partial\Omega} \frac{\partial u}{\partial n} u_t}$$
$$= 0$$

Some motivating applications:

(1) Longitudinal vibration of linear springs leads to a finite difference version of the

1D wave eqn



Let u_i = displacement of i^{th} node to the right (pos) or left (neg).

Assume (uniform) spring constants E + (uniform) masses (ρ at each node). Then

net force on node i = $E(u_{i+1}-u_i) - E(u_i - u_{i-1})$
accel of node i = $\ddot{u}_i$

So force = mass × accel becomes

$$\rho \ddot{u}_i = E\left(u_{i+1} + u_{i-1} - 2u_i\right)$$

ie a finite-difference version of the 1D wave eqn.

Note: if masses varied (mass i at node i) + spring constants varied (E_i on spring btwn x_i + x_{i+1}) same arg't would give

$$\rho_i \ddot{u}_i = E_i(u_{i+1} - u_i) - E_{i-1}(u_i - u_{i-1})$$

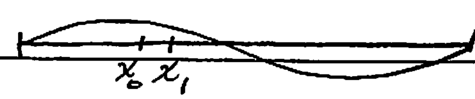
$\Rightarrow$ finite difference version of

$$\rho(x) u_{tt} = \text{div}\,(E(x)\nabla u) \quad \text{in 1D}$$

Using a nonlinear spring law, we could easily get a more nonlinear eqn.

(2) Transverse vibrations of a stretched string (eg violin string) or membrane (eg drum) also lead to wave eqns ($u_{tt} - c^2 u_{xx} = 0$ or $u_{tt} - c^2(u_{xx} + u_{yy}) = 0$) with some appropriate (linearizing) hypotheses.

Expln, following Strauss (see also Guenther + Lee §1.2): consider elastic string of length l, stretch in such a way that tension $T(x,t) > 0$. Set ρ = density (const) + $u = u(x,t)$ = vert displ.



Consider Newton's law (force = mass × accel) for the part of the string btwn x_0 + x_1. Noting that slope of string is u_x, we get

$$\left.\frac{T}{(1+u_x^2)^{1/2}}\right|_{x_2}^{x_1} = 0$$

LHS is net horizontal force, RHS = 0 since there's no horizontal motion

$$\left.\frac{T u_x}{\sqrt{1+u_x^2}}\right|_{x_0}^{x_1} = \int_{x_0}^{x_1} \rho\, u_{tt}\, dx$$

LHS is net vertical forces; RHS is vert component of accel.

Now assume small slope

$$(1+u_x^2)^{1/2} \sim 1 + \tfrac{1}{2} u_x^2$$

At leading order 1st eqn says T is indep of x. We then expect it's also indep of t, + 2nd eqn becomes

$$T u_{xx} = \rho\, u_{tt}$$

ie leading-order eqn in small-slope setting is

$$u_{tt} = c^2 u_{xx}\ , \quad c = \sqrt{T/\rho}$$

(we'll see soon that c = wave speed).

Variations on this:

- air resistance introduces an additional "damping" force in the vertical eqn, leading to $\rho u_{tt} - T u_{xx} + \gamma u_t = 0$

Note: if $\rho \to 0$ and $\gamma > 0$ this becomes a heat eqn. For $\rho > 0$ and $\gamma > 0$, the "kinetic + potential" energy decays:

$$\frac{d}{dt}\int_0^l \frac{\rho}{2}u_t^2 + \frac{T}{2}u_x^2 = \int_0^l \rho u_t u_{tt} + T u_x u_{xt}$$

$$= \int_0^l u_t(\rho u_{tt} - T u_{xx})\,dx + T u_x u_t \Big|_{x=0}^{x=l}$$

$$= -\gamma \int_0^l u_t^2$$ if bc makes the bdry term 0 (eg $u=0$ at ends, or $u_x = 0$ at ends)

- 2D case is very similar, provided state of stress is uniform tension T

For any region $D \subset$ membrane

$$\text{vert force on } D = \int_{\partial D} T\frac{\partial u}{\partial n} = \int_D \rho u_{tt}$$

So $$\int_D \operatorname{div}(T\nabla u) = \int_D \rho u_{tt}$$ for any region D

whence (if T is const) $\rho u_{tt} - T\Delta u = 0$, ie. $u_{tt} = c\Delta u$ ($c = \sqrt{T/\rho}$ as before).

(3) Other physical applns:

- <u>acoustic waves</u> (small-amplitude disturbances of approx-uniform pressure in an ideal gas), see Guenther + Lee §1.7

- <u>electromagnetic waves</u>: Maxwell's eqns, in simplest setting (no current, no charges) say

$$\frac{\varepsilon}{c} E_t = \operatorname{curl} H \qquad \operatorname{div} H = 0$$

$$\frac{\mu}{c} H_t = -\operatorname{curl} E \qquad \operatorname{div} E = 0.$$

If ε, μ, c are constant, then vector identity $\operatorname{curl}\operatorname{curl} H = \nabla \operatorname{div} H - \Delta H$ gives

$$\frac{\varepsilon}{c}(\operatorname{curl} E)_t = -\Delta H \implies \frac{\varepsilon\mu}{c^2} H_{tt} = \Delta H$$

$$\frac{\mu}{c}(\operatorname{curl} H)_t = \Delta E \implies \frac{\varepsilon\mu}{c^2} E_{tt} = \Delta E$$

so each component of H + E solves a scalar wave eqn with wavespeed $c/\sqrt{\varepsilon\mu}$. (See §12.1 of Guenther + Lee for a slightly more general discussion.)

Turning now to repn + properties of solns, we focus first on 1D wave eqn, which is easy + already interesting.

Focus first on the Cauchy problem

$$(*) \quad \begin{array}{ll} u_{tt} - u_{xx} = 0 & x \in \mathbb{R},\ t > 0 \\ u(x,0) = f(x) & \\ u_t(x,0) = g(x) & \end{array} \Big\} \text{ at } t=0$$

Notes:

a) unlike heat eqn we must specify both u and u_t at $t=0$

b) sign in eqn is crucial: $u_{tt} - u_{xx}$ is wave eqn, $u_{tt} + u_{xx} = 0$ would be a 2D Laplace eqn (for which specifn of both $u + u_t$ at $t=0$ is not possible!)

c) eqn $u_{tt} - c^2 u_{xx} = 0$ is easily reduced to $u_{tt} - u_{xx} = 0$ by linear chg of vars (if c is constant!). So I focus on $u_{tt} - u_{xx} = 0$ (to avoid clutter).

General 1D soln formula for (*):

$$u = F(x+t) + G(x-t)$$

Easy to check that if u has this form it satisfies eqn. For converse: let $\xi = x+t$, $\eta = x-t$. Then $x = \frac{1}{2}(\xi+\eta)$, $t = \frac{1}{2}(\xi-\eta)$, so $\varphi_\xi = \varphi_x \frac{\partial x}{\partial \xi} + \varphi_t \frac{\partial t}{\partial \xi}$ $\Rightarrow$

$$\varphi_\xi = \tfrac{1}{2}(\varphi_x + \varphi_t)$$
$$\varphi_\eta = \tfrac{1}{2}(\varphi_x - \varphi_t)$$

So $u_{tt} - u_{xx} = 0 \iff (\partial_t - \partial_x)(\partial_t + \partial_x)\, u = 0$

$$\iff -4\partial^2_{\xi\eta}\, u = 0$$

Now, $\partial^2_{\xi\eta} u = 0 \Rightarrow u = F(\xi) + G(\eta)$.

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In fact it's easy to read off $F + G$ from the initial data: at $t=0$ evidently

$$u(x,0) = F(x) + G(x) = f(x)$$
$$u_t(x,0) = F'(x) - G'(x) = g(x)$$

So $\begin{matrix} F' + G' = f' \\ F' - G' = g \end{matrix} \underset{\text{linear algebra}}{\Longrightarrow} F' = \frac{f'+g}{2}, \quad G' = \frac{f'-g}{2}$

Integrating:

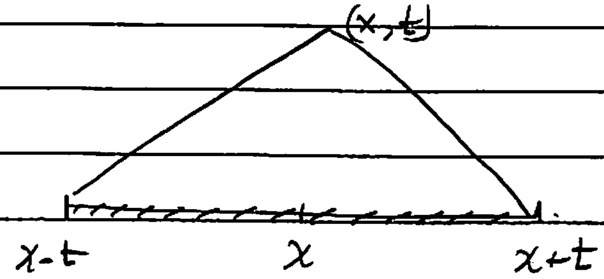
$$F(x) = \tfrac{1}{2} f(x) + \tfrac{1}{2}\int_0^x g(\xi)\,d\xi + c_0$$

$$G(x) = \tfrac{1}{2} f(x) - \tfrac{1}{2}\int_0^x g(\xi)\,d\xi - c_0$$

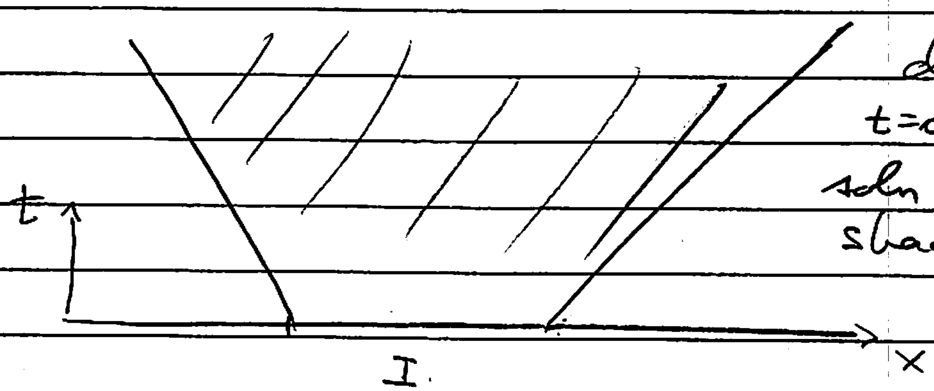
where c_0 is a constant of integr. (There is only one const of integr since $F+G=f$.) We can set $c_0=0$ since its value doesn't affect u. Then finally:

$$\boxed{\begin{aligned} u(x,t) &= F(x+t)+G(x-t) \\ &= \frac{1}{2}\left[f(x+t)+f(x-t)\right]+\frac{1}{2}\int_{x-t}^{x+t} g(\xi)\,d\xi . \end{aligned}}$$

Note phenomenon of domain of dependence:



$u(x,t)$ depends on the initial data only in the interval $(x-t,\ x+t)$



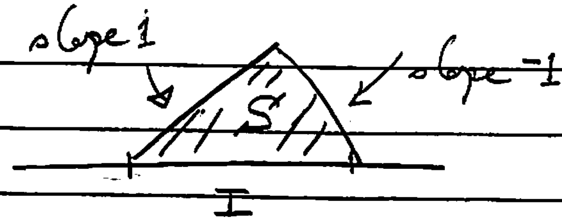
data in I at $t=0$ influence soln only in the shaded region

Consequences + notes:

- "information propagates at finite speed" (speed 1, for $u_{tt} - u_{xx} = 0$). Very different from heat eqn!
- soln is not smoother than its initial data
- eqn can be solved backward in time just as easily as forward in time.

For eqn $u_{tt} - c^2 u_{xx} = 0$ (c = constant) the story is essentially the same, except that $u = F(x+ct) + G(x-ct)$ and information propagates at speed c

There's an alternative "energy-based" pf of domain of dependence, which is important because it extends straightforwardly to higher dims (where soln formulas exist but are more complicated). Focus as before on $c=1$ ($u_{tt} - u_{xx} = 0$).



<u>Claim</u>: if $u = u_t = 0$ on I and $u_{tt} - u_{xx} = 0$ then $u \equiv 0$ everywhere in S (see figure).

<u>Pf</u>; May suppose (by replacing u with $u_\lambda(x,t) = u(\lambda x, \lambda t)$) that $I = (-1,+1)$. Consider the "energy" in each time slice

$$e(t) = \int_{-1+t}^{1-t} \tfrac{1}{2} u_t^2 + \tfrac{1}{2} u_x^2 \, dx$$

$$\frac{de}{dt} = \int_{-1+t}^{1-t} u_t u_{tt} + u_x u_{xt} \, dx \quad + \text{ bdry terms}$$

$$= \int_{-1+t}^{1-t} u_t \underbrace{(u_{tt} - u_{xx})}_{0} \, dx \quad + \text{ different bdry terms}$$

We could proceed to examine the bdry terms, but it's more efficient to <u>start over again</u>, looking for an appropriate "integr by parts"

Consider the vector field in the (x,t) plane:

$$\sigma = \left[-u_x u_t , \ \tfrac{1}{2} u_t^2 + \tfrac{1}{2} u_x^2 \right]$$

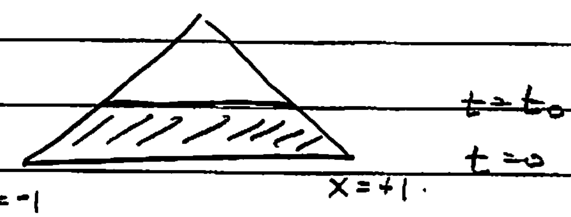
and note that

$$\operatorname{div}\sigma = \partial_t\left(\tfrac{1}{2}u_t^2 + \tfrac{1}{2}u_x^2\right) + \partial_x\left(-u_x u_t\right)$$

$$= u_t u_{tt} + \cancel{u_x u_{xt}} - u_t u_{xx} - \cancel{u_x u_{xt}}$$

$$= u_t(u_{tt} - u_{xx})$$

so our prev calcn can be redone using the 2D divergence theorem, applied to region between $t=0$ + $t=t_0$

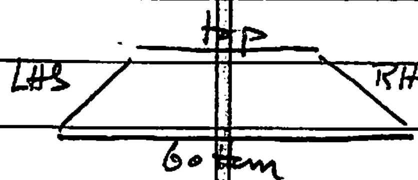


$$0 = \int_{\substack{\text{shaded}\\ \text{region}}} \operatorname{div}\sigma = \int_{\text{bdry}} \sigma\cdot n$$

$$= \int_{\text{top}} \tfrac{1}{2}(u_t^2 + u_x^2) - \int_{\text{bottom}} \tfrac{1}{2}(u_t^2 + u_x^2)$$

$$+ \frac{1}{\sqrt{2}}\int_{\text{RHS}} \left(\tfrac{1}{2}u_t^2 + \tfrac{1}{2}u_x^2 - u_x u_t\right) ds$$

$$+ \frac{1}{\sqrt{2}}\int_{\text{LHS}} \left(\tfrac{1}{2}u_t^2 + \tfrac{1}{2}u_x^2 + u_x u_t\right) ds$$



at RHS: $\vec{n} = \frac{1}{\sqrt{2}}(1,1)$

at LHS: $\vec{n} = \frac{1}{\sqrt{2}}(-1,1)$

The integrands on the LHS + RHS integrals are ≥ 0. So

$u_t = u_x = 0$ on initial interval $\Longrightarrow$

$$\int_{\text{top}} \tfrac{1}{2}(u_t^2 + u_x^2) + \int_{LHS} \tfrac{1}{2\sqrt{2}}(u_x + u_t)^2 + \int_{RHS} \tfrac{1}{2\sqrt{2}}(u_x - u_t)^2 = 0$$

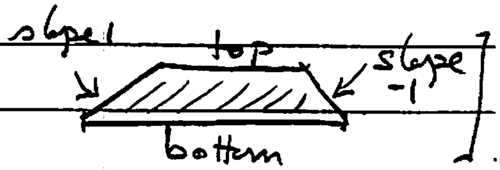
$\Rightarrow$ each term vanishes

$\Rightarrow$ $u_t = u_x = 0$ along the "top" (the segment $(-1+t_0, 1-t_0)$ at time t_0).

As t_0 varies btwn 0 + 1 we obtain the claim.

[Note: This argt also shows that for any choice of initial data,

$$\int_{\text{top}} u_t^2 + u_x^2 \leq \int_{\text{bottom}} u_t^2 + u_x^2$$



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What about bounded domains? For example

$$u_{tt} - u_{xx} = 0 \quad \text{for} \quad 0 < x < L.$$
$$u = 0 \quad \text{at} \quad x = 0, L.$$

Actually: it's just as easy to do $u_{tt} - \Delta u = 0$

in $\Omega \subset \mathbb{R}^n$ (bounded) with $u=0$ at $\partial\Omega$. (Neumann bc $\partial u/\partial n = 0$ is not fundamentally different.)

<u>Separation of vars</u> is a good tool here, using eigenfns of Δ with the chosen bc

$$u = \sum a_j(t)\, \varphi_j(x) \qquad -\Delta\varphi_j = \lambda_j \varphi_j \text{ in } \Omega$$

[and $\varphi_j = 0$ at $\partial\Omega$, if we want $u=0$ at $\partial\Omega$]

Coeffts must solve $\ddot{a}_j = -\lambda_j a_j$, so

$$a_j(t) = \alpha_j \cos\sqrt{\lambda_j}\, t + \beta_j \sin\sqrt{\lambda_j}\, t$$

(or, using complex notation: $a_j = \mathrm{Re}\left(c_j e^{i\sqrt{\lambda_j}\, t}\right)$).
Initial data determine $\alpha_j + \beta_j$.

A similar repn is possible for $u_{tt} = \Delta u$ on $\mathbb{R}^n$ using Fourier transform.

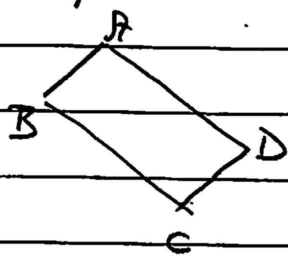
But: sep of vars soln <u>hides</u> the fact that information propagates at speed 1.

In 1D there's a different approach to

$$u_{tt} - u_{xx} = 0 \qquad 0 < x < L.$$
$$u = 0 \qquad \text{at } x = 0, L.$$
$$\left.\begin{aligned} u &= f(x) \\ u_t &= g(x) \end{aligned}\right\} \qquad \text{at } t = 0$$

which makes finite speed of propagation more evident. It uses

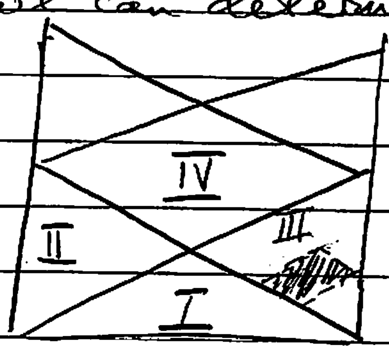
Key observation: for a parallelogram with sides of slope ± 1 in spacetime



$u_{tt} - u_{xx} = 0$ inside $\Rightarrow$ $u(A) - u(B) - u(D) + u(C) = 0$.

Pf: Recall that eqn says $u_{\xi\eta} = 0$ when $\xi = x+t$, $\eta = x-t$. In (ξ,η) plane our parallelogram is a rectangle. Assertion follows from elementary calculus.

Using this: we can determine u iteratively



- in region I, bdry has no effect
- in II + III use bdry for one vertex of parallelogram
- etc

For a half-line (with $u=0$ or $u_x=0$ at bdry) we can use reflection – same trick we used for the heat eqn:

- to solve $u_{tt} - u_{xx} = 0$ in $x>0$, $t>0$ with $u=0$ at $x=0$, look for a soln on all $\mathbb{R}$ that's <u>odd</u> in x (using odd reflection of initial data)
- if bc is instead $u_x = 0$ at $x=0$, look for soln on all $\mathbb{R}$ that's <u>even</u> in x (using even refln of initial data)

This also works for an interval:

- to solve $u_{tt} - u_{xx} = 0$ for $0<x<L$, $t>0$ with $u=0$ (or $u_x=0$) at endpts, use odd (or even) reflection to create $2L$-periodic initial data on real line then use 1D solution formula.