Goal for Lectures 9+10: linear wave eqn
\[ u_{tt} - \Delta u = 0 \] + related matters.

Essential character of this eqn (quite different from the heat eqn) is its "Hamiltonian" character: eqn is a PDE version of "force = mass * accel". Also the (related) fact that "kinetic + potential energy" is conserved, e.g.

if \[ u_{tt} \big|_{t=0} = 0 \quad \text{for} \quad x \in \mathbb{R} \]
\[ u = 0 \quad \text{at} \quad t = \Delta t \]

Then \( \frac{\partial}{\partial t} \frac{1}{2} \int_{\mathbb{R}} u^2 + \frac{1}{2} \Delta u^2 \, dx \)

\[ = \int_{\mathbb{R}} u u_{tt} + \nabla u \cdot \nabla u \, dx \]
\[ = \int_{\mathbb{R}} (u_{tt} - \Delta u) + \frac{\partial}{\partial x} \left( \frac{\partial u}{\partial x} \right) \, dx \]
\[ = 0 \]

Some motivating applications:

1. Longitudinal vibration of linear spring leads to a finite difference version of this.
1D wave eqn

Let $u_i =$ displacement of $i^{th}$ node to the right (pos) or left (neg).

Assume (uniform) spring constants $F$ and (uniform) mass $m$ at each node. Then

net force on node $i = E(u_i - u_{i-1}) - E(2u_i - u_{i+1})$

accel of node $i = \ddot{u}_i$

So force = mass $\times$ accel becomes

$\rho \ddot{u}_i = E(u_i - u_{i-1} - 2u_i)$

is a finite-difference version of the 1D wave eqn.

Note: if masses varied (mass $i$ at node $i$) + spring constants varied (e.g., $x_i$ + $x_{i+1}$) same not would work.

$\rho_i \ddot{u}_i = E_i(u_i - u_{i-1}) - E_{i+1}(2u_i - 2u_{i+1})$
finite difference version of

\[ \Delta u_{tt} = \nabla \cdot (\nabla u) \]

Using a nonlinear spring law, one could easily get a more nonlinear eqn.

(2) Transverse vibrations of a stretched string (eg violin string) or membrane (eg drum) also lead to wave eqns \( u_{tt} - c^2 u_{xx} = 0 \) or \( u_{tt} - c^2 (u_{xx} + u_{yy}) = 0 \) with some appropriate (linearizing) hypotheses.

Explore, following Strauss (see also Gunther + Lee 2011): consider elastic string of length \( l \) stretched in such a way that tension \( T > 0 \). Set \( p = \text{density (const.)} + \rho = \rho(x,t) = \text{vert. deflection} \).

Consider Newton's law \( \text{force} = \text{mass} \times \text{accel} \) for the part of the string between \( x_0 \) and \( x_1 \). Noting that slope of string is \( u_x \), we get

\[ \frac{T}{(1 + u_x^2)^{1/2}} \left| \begin{array}{c} x_1 \\ x_2 \end{array} \right| = 0 \]

LHS is not horizontal

since, \( \text{RHS} = 0 \) since

there's no horizontal motion
\[ \frac{T_{u_x}}{\sqrt{1 + u_x^2}} = \int_{x_0}^{x_1} u_{tt} \, dx \]

LHS is net vertical force
RHS is vertical component of accel.

New assume small slope

\[ (1 + u_x^2)^{-\frac{1}{2}} \approx 1 + \frac{1}{2} u_x^2 \]

At leading order 1st eqn says \( T \) is uphill of \( x \).
We then expect it's also uphill of \( t \), 2nd eqn becomes

\[ T_{u_{xx}} = \frac{1}{2} u_{tt} \]

In leading-order eqn in small-slope setting is

\[ u_{tt} = c^2 u_{xx} \quad c = \sqrt{\frac{1}{\rho}} \]

(we'll see soon that \( c = \) wave speed).

Variations on this:

- air resistance introduces an additional "damping" force on the vertical \( y \) \( x \) leading to \[ \rho u_{tt} - T_{u_{xx}} + \delta u_t = 0 \]

Note: if \( \rho \to 0 \) and \( \delta > 0 \) this becomes a
For $p > 0$ and $x > 0$, the "kinetic + potential" energy decay is:

$$
\int_0^l \left( \frac{1}{2} u_t^2 + \frac{1}{2} u_x^2 \right) \, dx = \int_0^l \left( \frac{1}{2} u_{tt} - T u_x \right) \, dx
$$

For $x = 0$ and $x = l$ the boundary terms are:

$$
\left. \frac{1}{2} u_t^2 \right|_{x=0} = \left. \frac{1}{2} u_t^2 \right|_{x=l}
$$

The 2D case is very similar, provided state of stress is uniform tension $T$.

For any region DC membrane

$$
\text{vertical force on } D = \iint_D T \frac{\partial u}{\partial n} \, dA = \iint_D \frac{\partial u}{\partial t} \, dA
$$

So

$$
\int \text{div} (T \nabla u) = \iint_D \frac{\partial u}{\partial t} \, dA
$$

for any region $D$.

Where (if $T$ is constant)

$$
\frac{\partial u}{\partial t} - T \Delta u = 0
$$

is $u_{tt} = c^2 \Delta u$ (c = $\sqrt{T/\rho}$ as before).
(3) Other physical applications:

- **acoustic waves** (small-amplitude disturbances of approx-uniform pressure in an ideal gas), see Guenther + Lee 31.7

- **electromagnetic waves**: Maxwell's equations in simplest setting (no current, no charges) say

  \[
  \frac{\mathbf{E}}{c} \times \frac{\mathbf{H}}{c} = \nabla \times \mathbf{H} \quad \text{div} \mathbf{H} = 0
  \]

  \[
  \frac{\mathbf{H}}{c} \times \frac{\mathbf{E}}{c} = -\nabla \times \mathbf{E} \quad \text{div} \mathbf{E} = 0.
  \]

  If \( \mathbf{E}, \mathbf{H} \) are constant, then vector identity \( \nabla \times (\nabla \times \mathbf{H}) = \nabla (\nabla \cdot \mathbf{H}) - \Delta \mathbf{H} \) gives

  \[
  \frac{\mathbf{E}}{c} \times (\nabla \times \mathbf{H}) \left( \frac{\partial}{\partial t} \right) = -\Delta \mathbf{H} \Rightarrow \frac{\mathbf{E}}{c^2} \frac{\partial^2 \mathbf{H}}{\partial t^2} = \Delta \mathbf{H}
  \]

  \[
  \frac{\mathbf{H}}{c} \times (\nabla \times \mathbf{E}) \left( \frac{\partial}{\partial t} \right) = \Delta \mathbf{E} \Rightarrow \frac{\mathbf{E}}{c^2} \frac{\partial^2 \mathbf{E}}{\partial t^2} = \Delta \mathbf{E}
  \]

  so each component of \( \mathbf{H} + \mathbf{E} \) solves a scalar wave eqn with wave speed \( \frac{c}{\sqrt{\mu \epsilon}} \). (See 3.12.1 of Guenther + Lee for a slightly more general discussion.)
Turning now to wave properties of solns, we focus first on 1D wave eqn. which is easy and already interesting.

Focus first on the Cauchy problem

\[ u_{tt} - u_{xx} = 0 \quad x \in \mathbb{R}, \ t > 0 \]
\[ u(x,0) = f(x) \quad x \in \mathbb{R}, \ t = 0 \]
\[ u_t(x,0) = g(x) \quad x \in \mathbb{R}, \ t = 0 \]

Notes:

a) Unlike heat eqn we must specify both \( u \) and \( u_t \) at \( t = 0 \)

b) Sign in eqn is crucial; \( u_{tt} - u_{xx} \) in wave eqn, \( u_{tt} + u_{xx} = 0 \) would be a 2D Laplace eqn (for which \( \text{speed of both } u_t \) and \( u_{xx} \) at \( t = 0 \) is not possible !)

c) Eqn \( u_{tt} - c^2 u_{xx} = 0 \) is easily reduced to \( u_{tt} - u_{xx} = 0 \) by linear change of vars (if \( c \) is constant ?). So I focus on \( u_{tt} - u_{xx} = 0 \) (to avoid clutter).

General 1D soln formula for \( u \):

\[ u = F(x+t) + G(x-t) \]
Easy to check that it has this form if
satisfies eqn. For converse: let \( \xi = x + t, \eta = x - t \).
Then \( x = \frac{1}{2}(\xi + \eta), t = \frac{1}{2}(\xi - \eta) \), so
\( \phi_x = \frac{\partial}{\partial \xi} (\phi_x + \phi_\eta), \quad \phi_\eta = \frac{\partial}{\partial \eta} (\phi_x - \phi_\eta) \)

\[ \phi_x = \frac{1}{2} (\phi_x + \phi_\eta) \]
\[ \phi_\eta = \frac{1}{2} (\phi_x - \phi_\eta) \]

So \( \frac{\partial}{\partial t} - \frac{\partial}{\partial x} = 0 \iff \left( \frac{\partial}{\partial t} - \frac{\partial}{\partial x} \right) \phi = 0 \]
\[ \iff \xi_d^2 \phi = 0 \]

Now, \( \partial_{\xi \eta}^2 \phi = 0 \Rightarrow \phi = F(\xi) + G(\eta) \).

In fact it's easy to read off \( F + G \) from the initial data: at \( t = 0 \) evidently

\[ u(x, 0) = F(x) + G(x) = \tilde{f}(x) \]
\[ u_t(x, 0) = F'(x) - G'(x) = \tilde{g}(x) \]

So \( F' + G' = \tilde{f}' \Rightarrow F' = \frac{\tilde{f}' + \tilde{g}}{2}, \quad G' = \frac{\tilde{f}' - \tilde{g}}{2} \)

Integrate:

\[ F(x) = \frac{1}{2} \tilde{f}(x) + \int_0^x \tilde{g}(\xi) d\xi + c \]
\[ G(x) = \frac{1}{2} \tilde{f}(x) - \int_0^x \tilde{g}(\xi) d\xi - c \]
where $c_0$ is a constant of integration (There is only one such constant since $F + G = f$.) We can set $c_0 = 0$ since its value doesn't affect $u$.

Thus, finally:

$$u(x, t) = f(x + t) + g(x - t)$$

$$= \frac{1}{2} \left[ f(x + t) + f(x - t) \right] + \frac{1}{2} \int_{x - t}^{x + t} g(s) \, ds.$$ 

Note the phenomenon of **domain of dependence**:

$u(x, t)$ depends on the initial data only in the interval $(x - t, x + t)$.
Consequences + notes:

- "Information propagates at finite speed" (speed 1, for \( \frac{\partial}{\partial t} - \frac{\partial}{\partial x} = 0 \)). Very different from heat eq!
- Solution is not smoother than its initial data.
- Eqn can be solved backward in time just as easily as forward in time.

For eqn \( \frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = 0 \) (\( c \) = constant) the story
is essentially the same, except that
\( u = F(x+ct) + G(x-ct) \) and information propagates
at speed \( c \).

There's an alternative "energy-based" pf of
domain of dependence, which is important because
it extends straightforwardly to higher chars
(while soln formulas exist but are more
complicated). Focus as before on \( c = 1 \)
\( \frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = 0 \).
Claim: if \( u = v = 0 \) on \( \Gamma \) and \( u_t + 2uv_x = 0 \) then \( u = 0 \) everywhere in \( S \) (see figure).

**Proof:** May suppose (by replacing \( u \) with \( u(x, t) = u(2x, 2t) \)) that \( \Gamma = (-1, 1) \). Consider the "energy" in each time slice

\[
e(t) = \int_{-1+t}^{1-t} \left( \frac{1}{2} u_t^2 + \frac{1}{2} u_x^2 \right) dx
\]

\[
\frac{d}{dt} e(t) = \int_{-1+t}^{1-t} u_t u_x + u_x u_t dx + \text{body terms}
\]

\[
= \int_{-1+t}^{1-t} u_t (2u_t + 2u_{xx}) dx + \text{body terms}
\]

We could proceed to examine the body terms, but it's more efficient to start over again, looking for an appropriate "integro by parts."

Consider the vector field in the \((x, t)\) plane:

\[
\mathbf{v} = [-u_x u_t + \frac{1}{2} u_t^2 + \frac{1}{2} u_x^2]
\]
and note that

\[ \partial_t \sigma = \partial_t \left( \frac{1}{2} u_t^2 + \frac{1}{2} u_x^2 \right) + \partial_x \left( -u_x u_t \right) \]

\[ = u_t u_{tt} + u_{xx} u_t - u_t u_{xx} - u_{xx} u_t \]

\[ = u_t (u_{tt} - u_{xx}) \]

So our proofcalc can be rendered using the 2D divergence theorem, applied to region between
\[ t = 0 + \quad t = t_0 \]

\[ x = -1 \quad x = +1 \quad t = t_0 \]

\[ \sigma = \int_{\text{shaded}} d\sigma = \int_{\text{body}} d\sigma - \int_{\text{region}} d\sigma \]

\[ = \int_0^{t_0} \left[ \frac{1}{2} u_t^2 + \frac{1}{2} u_x^2 \right] - \int_0^{t_0} \left[ \frac{1}{2} (u_t^2 + u_x^2) \right] \]

\[ + \frac{1}{12} \int \left( \frac{1}{2} u_t^2 + \frac{1}{2} u_x^2 - u_x u_t \right) \, dA \]

\[ \text{RHS} \]

\[ \int \left( \frac{1}{2} u_t^2 + \frac{1}{2} u_x^2 + u_x u_t \right) \, dA \]

\[ \text{LHS} \]

at \text{RHS:} \quad \sigma = \frac{1}{12} (111)

at \text{LHS:} \quad \sigma = \frac{1}{12} (-111)
The integrands on the LHS + RHS integrals are ≥ 0. So

\[ u + u_t = 0 \text{ on initial interval } \Rightarrow \]

\[ \int_{\text{top}}^{\text{LHS}} \left( \frac{1}{2} u_t^2 + u_x^2 \right) \, dx + \int_{\text{top}}^{\text{RHS}} \frac{1}{2} (u_x + u_t)^2 \, dx + \int_{\text{top}}^{\text{top}} \frac{1}{2} (u - u_t)^2 \, dx = 0 \]

\Rightarrow \text{each term vanishes}

\Rightarrow u_t = u_x = 0 \text{ along the "top" (the segment } (-1 + t_0, 1 - t_0) \text{ at } t = t_0). \]

As \( t \) varies from 0 to 1 we obtain the claim.

[Note: This argument also shows that for any choice of initial data,

\[ \int_{\text{top}}^{\text{top}} u_t^2 + u_x^2 \leq \int_{\text{bottom}}^{\text{bottom}} u_t^2 + u_x^2 \]

What about bounded domains? For example

\[ u_{tt} - u_{xx} = 0 \text{ for } 0 < x < L. \]

\[ u = 0 \text{ at } x = 0, L. \]

Actually: it's just as easy to do \( u_{tt} - \Delta u = 0 \)
in $\mathbb{R}^n$ (bounded) with $u=0$ at $\partial \Omega$. (Neumann be $\partial u/\partial n = 0$ is not fundamentally different.)

Separation of rows is a good tool here, using eigenvalues $\lambda$ with the chosen $\phi_j$ be

$$u = \sum_j a_j(t) \phi_j(x) \text{,} \quad -\Delta \phi_j = \lambda \phi_j \text{ in } \Omega$$

[and $\phi_j = 0 \text{ at } \partial \Omega$, if we want $u=0$ at $\partial \Omega$.]

Coefficients must satisfy $\frac{\partial a_j}{\partial t} = -\Delta \phi_j$, i.e.,

$$a_j(t) = a_j(0) e^{-\lambda t} + \beta_j \sin \sqrt{\lambda} t$$

(or, using complex notation: $a_j = \Re (c_j e^{i\sqrt{\lambda} t})$).

Initial data determine $\alpha_j + \beta_j$.

A similar result is possible for $\Delta_t = \Delta x$ in $\mathbb{R}^n$ using Fourier transform.

Best: separation of rows hides the fact that information propagates at speed 1.

In 1D, there's a different approach to
\[ u - u_x = 0 \quad 0 < x < L. \]
\[ \frac{\partial u}{\partial t} = 0 \quad \text{at } x=0, L. \]
\[ u = f(x) \quad \text{at } t=0. \]
\[ u = g(x) \quad \text{at } t=0. \]

which makes finite speed of propagation more evident. It uses

Key observation: for a parallelogram with sides of slope ±1 in spacetime

\[ u_{tt} - u_{xx} = 0 \quad \text{inside} \quad \Rightarrow u(A) - u(B) - u(D) + u(C) = 0. \]

Recall that one says \( u = 0 \) when \( \xi = x + t, \eta = x - t \). In \((\xi, \eta)\) plane our parallelogram is a rectangle. Assertion follows from elementary calculus.

Using this: we can determine \( u \) iteratively.

- in region I, \( u \) only has no effect.
- in II + III use \( u \) only for one vertex of parallelogram.
- etc.
For a half-line (with \( u = 0 \) or \( u_x = 0 \) at \( x = 0 \)), we can use reflection - same trick we used for the heat eqn:

- to solve \( u_t - u_{xx} = 0 \) in \( x > 0, t > 0 \)
  - with \( u = 0 \) at \( x = 0 \), look for a solution \( u \) on all \( \mathbb{R} \) that's odd in \( x \) (using odd reflection of initial data)

- if \( \partial_x u \) instead \( u_x = 0 \) at \( x = 0 \),
  - look for a solution \( u \) on all \( \mathbb{R} \) that's even in \( x \) (using even reflection of initial data)

This also works for an interval:

- to solve \( u_{tt} - u_{xx} = 0 \) in \( 0 < x < l, t > 0 \)
  - with \( u = 0 \) (or \( u_x = 0 \)) at ends to,
  - use odd (or even) reflection to create 2\( l \)-periodic initial data on real line then use 1D solution formula.