PDE - Lecture 8, 11/4/2014

[start with "Lecture 7 notes," which we didn't get to last week]

Last "elliptic" topic: the "big picture" concerning several approaches to existence, then some detail concerning variational methods. I'll focus mainly on $\Delta u = 0$ in Ω, $u|_{\partial\Omega} = g$ at $\partial\Omega$ but the story for Neumann bc is similar (and eqns of the form $\Delta u = f$ in Ω are easily reduced to the homog pblm $\Delta\tilde{u} = 0$ if f extends to a smooth fn on all $\mathbb{R}^n$, by taking $\tilde{u} = u - \Phi * f$ and adjusting the bc accordingly).

Note: it isn't suff't to "just use the Green's fn," since our argument for the *existence* of the Green's fn assumed you can solve $\Delta u = 0$ in Ω, $u|_{\partial\Omega} = g$ for nice ~~enough~~ g.

There are 3 main approaches to existence:

(1) *Perron's method* rests on the observations that

- if $\Delta w \geq 0$ in Ω ("w is subharmonic") and $w \leq u$ at $\partial\Omega$ then $w \leq u$ in Ω (by max prin, since $\Delta(w-u) \geq 0$).

• same stmt even if w is merely cont's and "$\Delta w \geq 0$" is replaced by

$$(*) \quad w(x) \leq \frac{1}{|\partial B_r(x)|} \int_{|y-x|=r} w(y)\, dA_y$$

(since pf of max prin uses only mean value principle; note by Prob Set 6 #1 that $\Delta w \geq 0$ does imply $(*)$).

So: soln can be characterized as

u = largest cont's fn w s.t. $w \leq g$ at $\partial\Omega$ and w is subharmonic (in sense of 2nd bullet)

See eg Fritz John's book 3.4.4 for pf that there is such a function, and that it's harmonic, with $u = g$ at $\partial\Omega$.

Advantage of this approach: it permits very general bc and rather singular domains (though some hypothesis is needed!).

Disadvantage: not well-suited to numerical implementation.

(2) Boundary integral method: solve an integral eqn on $\partial\Omega$; eg to solve $\Delta u = 0$ in Ω with $u = g$ at $\partial\Omega$, we can look for $\varphi : \partial\Omega \to \mathbb{R}$ st the fn u_φ defined (in $\mathbb{R}^n$, $n \geq 3$) by

$$u_\varphi(x) = \int_{\partial\Omega} \frac{\partial}{\partial \nu_y} \left(\frac{1}{|x-y|^{n-2}} \right) \varphi(y) \, dA_y$$

(which is definitely harmonic in Ω) has the desired bc at $\partial\Omega$. This is called the representation of u by a "double layer potential."

See Guenther + Lee §8.6–8.7 for a good treatment of this topic.

Advantage: works well numerically (in fact it is the method of choice for unbounded Ω, when discretizing space is impractical)

Disadvantage: method is rather special to Laplace's eqn, and relies heavily on linearity of the pde.

(3) Variational principles, eg

(a) $\Delta u = f$ in Ω, $u = g$ at $\partial\Omega$ $\iff$ $\displaystyle\min_{\substack{u=g \\ \text{at } \partial\Omega}} \int_\Omega \tfrac{1}{2}|\nabla u|^2 + fu \, dx$

(b) $\Delta u = f$ in Ω. $\dfrac{\partial u}{\partial n} = g$ at $\partial\Omega$. $\iff$ $\displaystyle\min_u \int_\Omega \tfrac{1}{2}|\nabla u|^2 + fu \, dx - \int_{\partial\Omega} ug \, dA$.

[assuming consistency]

Advantages: easy to implement numerically (by minimizing over a finite-dim'l class of u's), and extends easily to many nonlinear problems.

Disadvantages: not good for exterior domains. Also: not every pde comes from a var'l pbm. [But: for lin eqns there's a fix — known as the Lax-Milgram Lemma — that permits even eqns not assoc to var'l pbms to be reduced to a linear alg pbm similar to the one assoc to a var'l pbm).

Let's go a bit further on var'l principles:

- Fact that the min is achieved is nontrivial; for the Neumann pbm (b) for example it requires that data be consistent.

- Convexity wrto ∇u is also important here; for example

$$\min_u \int_0^1 (u_x^2-1)^2 + u^2 \, dx$$

does not achieve its min (the min is 0, but no function can have $u^2=0$ and $(u_x^2-1)^2=0$)

- if $W(\nabla u)$ is a strictly convex fn of ∇u, then it's easy to see that a suff'ly smooth soln of

$$\operatorname{div}\left(\frac{\partial W}{\partial \nabla u}\right) = 0 \text{ in } \Omega, \quad u=g \text{ at } \partial\Omega$$

achieves $\displaystyle \min_{\substack{u=g \\ \text{at } \partial\Omega}} \int_\Omega W(\nabla u) \, dx$

(we'll have a HW pbm on this; note that for $W(\nabla u) = |\nabla u|^2$ it was on Problem Set 5).

Rigorous treatment of existence by this method is PDE II material (but you can find it in Chapter 11 of Guenther + Lee, for example).

But let's discuss why variational principles like (a) + (b) above lead to successful numerical schemes, focusing for simplicity on

$$\Delta u = f \text{ in } \Omega \quad , \quad u = 0 \text{ at } \partial\Omega .$$

Numerically, we could

$\ast$ minimize $\int_\Omega \frac{1}{2}|\nabla u|^2 + fu$ over some finite-dimensional space S of functions on Ω that all vanish at $\partial\Omega$.

The condition of optimality is that opt'l u_S satisfies

$$(\ast\ast) \quad u_S \in S \text{ and } \int_\Omega \langle \nabla u_S, \nabla \varphi \rangle + f\varphi = 0 \text{ for all } \varphi \in S$$

which amounts to a finite-dimensional linear algebra problem.

Typical choices of S

- span of 1^{st} N basis fns from a well-chosen basis (eg eigenfns of the Laplacian, if known, eg in a rectangle; or perhaps a wavelet basis, down to a specified level of fine-ness)

or

- piecewise linear fns on a specified triangulation (this is the simplest example of a finite element method). Note that a piecewise linear fn is entirely determined by its nodal values; also, there are no restrictions on the nodal values (a piecewise linear fn is automatically cont's across edges)

How well does this work? Most basic result is that its performance is limited only by how well the true soln can be approximated in S; in fact if u_S solves (**) & u solves pde then

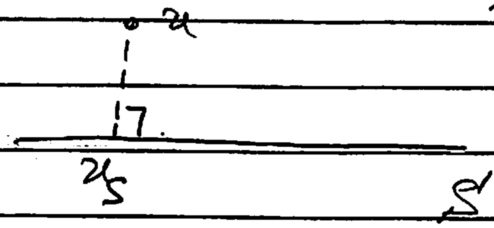
$$\int_\Omega |\nabla(u-u_S)|^2 = \min_{v\in S} \int_\Omega |\nabla(u-v)|^2$$

Proof of this: From 1st order optimality condn for the var'l principle + its restrn to S, we get

$$\int_\Omega \langle \nabla(u_S-u), \nabla\varphi \rangle = 0 \quad \text{for all } \varphi\in S.$$

This means u_S = orthog projn of u onto S w.r.to

The inner product $(u, v) = \int_\Omega (\nabla u, \nabla v)$.



Thus, from linear algebra, u_S is the closest pt in S to u. Done!

[Here's a direct version of the argument: for any $v \in S$ write $u - v = (u - u_S) + (u_S - v)$. Then

$$\int_\Omega \tfrac{1}{2} |\nabla(u-v)|^2 = \int_\Omega \tfrac{1}{2} |\nabla(u-u_S)|^2 + \int_\Omega \tfrac{1}{2}|\nabla(u_S - v)|^2$$

$$+ \int \langle \nabla(u-u_S), \nabla(u_S - v) \rangle.$$

Last term vanishes since $u_S - v \in S'$. So

$$\int_\Omega |\nabla(u-v)|^2 \geq \int |\nabla(u-u_S)|^2.$$

with equality only when $\int_\Omega |\nabla(u_S - v)|^2 = 0$, i.e. $u_S = v$ (using the bc).

For more on this topic see §8.5 of Strauss and §11.5 of Guenther + Lee (for short discussions) and 1st 50 pp of G Strang + G. Fix, "An analysis of the finite element method" (a very readable tutorial with much more detail.)