Last "elliptic" topic: the "big picture" concerning several approaches to existence. Then more detail concerning variational methods. I'll focus mainly on \( \Delta u = 0 \) in \( \Omega \), \( u \mid = g \) on \( \partial \Omega \) but the story for Neumann BC is similar (and see if the term \( \Delta u \) in \( \Omega \) are easily reduced to the homogeneous \( \Delta u = 0 \) if \( f \) extends to a smooth \( f \) on all \( \mathbb{R}^n \) by taking \( \tilde{u} = u - \frac{1}{n+2} \Delta \) and adjusting the BC accordingly).

Note: it can't hurt to "just use the Green's fn", since our agreement for the existence of the Green's fn assumed you can solve \( \Delta u = 0 \) in \( \Omega \), \( u \mid = g \) on \( \partial \Omega \) for nice enough \( f \).

There are 3 main approaches to existence:

(1) Perron's method rests on the observation that

- if \( \Delta W \geq 0 \) in \( \Omega \) ("W is subharmonic")
  and \( W \leq u \) at \( \partial \Omega \) then \( W \leq u \) in \( \Omega \)
  (by max prin, since \( \Delta (W-u) \geq 0 \)).
Same sort even if \( W \) is merely continuous and "\( \Delta W \approx 0 \)" is replaced by

\[
W(x) \leq \frac{1}{T(x)} \int_{y-x=\epsilon} W(y) \, dA
\]

(since \( \Delta \) of wax prior uses only mean value principle, note by Prob 1 in #1 that \( \Delta W \approx 0 \) does imply \( \approx \)).

So: \( \nu \) can be characterized as

\[ \nu = \text{largest cent's for } W - \epsilon \text{ at } \Omega \]

and \( W \) is subharmonic (in sense of 2nd bullet).

See eg. Fritz John's book 3.4.4 for pt that there is such a function, and that it's harmonic, with \( z = \epsilon \) at \( \Omega \).

Advantage of this approach: it permits very general \( \Omega \) and rather singular domains (though some hypothesis is needed!)

Disadvantage: not well-suited to numerical implementation.
(2) Boundary integral method: solve an integral eqn on \( \partial D \); e.g., to solve \( \Delta u = 0 \) in \( D \) with \( u=g \) on \( \partial D \), we can look for \( \varphi: \partial D \rightarrow \mathbb{R} \) at the "far" points defined (on \( \partial D \), \( n \geq 2 \)) by

\[
\varphi(x) = \int \frac{\partial}{\partial n} \left( \frac{1}{|x-y|^{n-2}} \right) \varphi(y) \, dS
\]

(which is definitely harmonic in \( D \)) to the desired bc. at \( \partial D \). This is called the representation of \( u \) by a "double layer potential."

See Grenander + Lee 3.8.6 - 3.7 for a good treatment of this topic.

Advantage: works well numerically (in fact, it is the method of choice for unbounded \( D \), when discretizing space is impractical)

Disadvantage: method is rather special to Laplace's eqn, and relies heavily on linearity of the pole.

(3) Variational principles, e.g.
\begin{align*}
\text{(a)} \quad \nabla u &= f \quad \text{on} \quad \overline{\Omega} \\
\implies \quad \min \int \frac{1}{2} \nabla u^2 + f u \, dx &\quad \text{on} \quad \overline{\Omega} \\
\nabla u &= f \\
\quad \text{on} \quad \partial \Omega.
\end{align*}

\begin{align*}
\text{(b)} \quad \Delta u &= f \quad \text{in} \quad \Omega \\
\implies \quad \min \int \frac{1}{2} \nabla u^2 + f u \, dx &\quad \text{in} \quad \Omega \\

\text{[Assuming consistency]}
\end{align*}

Advantages: easy to implement numerically (by minimizing over a finite-dimensional class of \( u \)), and extends easily to many nonlinear problems.

Disadvantages: not good for exterior domains, and not every problem comes from a variational problem. (But: some linear problems have a fix — known as the Lax-Milgram lemma — that permits even linear equations not arising from variational problems to be reduced to a linear algebra problem similar to the one above to a variational problem.)

Let's go a bit further on variational principles:
The fact that the min is achieved in nontrivial cases for the Neumann plan (6) for example it requires that data be consistent.

Conversely, w to 7r is also important here: for example,

\[ \min_u \int_0^1 \frac{(u_x^2 - 1)^2 + u^2}{u} \, dx \]

does not achieve its min. The min is 0, but no function can have \( u \to 0 \) and \( (u_x^2 - 1)^2 \to 0 \).

If \( W(\Omega) \) is a convex set of \( \Omega \), then it's easy to see that a suitable, weak solution of

\[ \text{div} \left( \frac{\partial W}{\partial u} \right) = 0 \quad \text{in} \, \Omega, \quad u = g \quad \text{at} \, \partial \Omega \]

achieves \( \min_u \int_\Omega W(\Omega) \, dx \)

\[ u = g \quad \text{at} \, \partial \Omega \]

(we'll leave a HW plan on this; note that for \( W(\Omega) = 13u^2 \) it was on Problem Set 5).

Reproposed treatment of existence by this method is PDE II material (but you can find it in Chapter 11 of Greenberg+Lee, for example).
But let's discuss why variational principles like (3) above lead to successful numerical schemes, focusing for simplicity on

\[ \text{Du} = f \text{ on } \Omega, \quad u = 0 \text{ on } \partial \Omega. \]

Numerically, we could

\[ \text{minimize } \int_{\Omega} \frac{1}{2} |\nabla u|^2 + f u \text{ over } u. \]

finite-dimensional space \( S \) of functions on \( \Omega \) that all vanish at \( \partial \Omega \).

The condition of optimality is that \( u_S \) satisfies

\[ (\forall \psi \in S \text{ and } \int_S \langle \nabla u, \nabla \psi \rangle + f \psi = 0 \text{ for all } \psi \in S, \]

which amounts to a finite-dimensional linear algebra problem.

Typical choices of \( S \):

- span of \( N \) basis functions from a well-chosen basis (e.g., eigenfunctions of the Laplacian, if known, or in a rectangle, or perhaps a wavelet basis), down to a specified
The level of fineness

- Piecewise linear basis on a specified triangulation (this is the simplest example of a finite element method). Note that a piecewise linear basis is determined by its nodal values; also, there are no restrictions on the nodal values (a piecewise linear basis is automatically continuous across edges).

How well does this work? Most basic result is that its performance is limited only by how well the true solution can be approximated in $S$. In fact if $v$ solves $(\mathbf{\star \star})$ with $p$ then

$$\left\| \mathbf{J}(v-u) \right\|^2 = \min_{v' \in S} \left\| \mathbf{J}(v-v') \right\|^2$$

Proof of this: From 1st order optimality conditions for the weak principle + its restriction to $S$, we get

$$\left\langle \mathbf{J}(v-u), v' \right\rangle = 0 \text{ for all } v' \in S.$$

This means $v$ is exactly projection of $u$ onto $S$. 
The inner product \( (u, v) = \frac{1}{n} \sum_{i=1}^{n} u_i v_i \).

Then, from linear algebra, \( u^* \) is the closest pt in \( S \) to \( u \). Done!

Here's a direct version of the argument:

For any \( v \in S \) write \( u - v = (u - u^*_S) + (u^*_S - v) \). Then

\[
\frac{1}{2} \left\| u - v \right\|^2 = \frac{1}{2} \left\| u - u^*_S \right\|^2 + \frac{1}{2} \left\| u^*_S - v \right\|^2 + \frac{1}{2} \left\langle u - u^*_S, u^*_S - v \right\rangle.
\]

Last term vanishes since \( u^*_S - v \in S \).

\[
\frac{1}{2} \left\| u - v \right\|^2 \geq \frac{1}{2} \left\| u - u^*_S \right\|^2.
\]

with equality only when \( \left\langle u - u^*_S, u^*_S - v \right\rangle = 0 \), i.e. \( u = v \) (using the 6th).

For more on this topic see 3.8.5 of Strauss and 3.11.5 of Guenther & Lee. For short discussions and 13-15 pp of G. Stone & G. Fix, "An analysis of the finite element method" (a very readable tutorial with much more detail).