PDE - Lecture 7, 10/28/2014

Continuing with Laplace-type eqns.

Major topics in Lecture 5 + 6 notes were

- examples + motivations
- uniquess (improved on Prob Set 6 to proofs of well-posedness) in bounded domains
- mean value property, with applns ($\Delta u = 0$ in $\mathbb{R}^n$ + sublinear growth at $\infty \Rightarrow u = \text{const}$; $\Delta u = 0 \Rightarrow u$ is C^∞)
- soln formula to $\Delta u = f$ in $\mathbb{R}^n$ (with sublinear growth at ∞): The Fundamental soln
- weak + strong forms of max prin
- Poisson's kernel, for a ball in $\mathbb{R}^2$.

This is a lot, but it falls short by giving us

a) no representation for solns of $\Delta u = f$ in bdd $\Omega \subset \mathbb{R}^n$ (with bc such as $u = g$ at $\partial\Omega$), except in special cases such as a ball $\subset \mathbb{R}^2$

b) no suggestion how solns could be found numerically (except by finite differences)

These notes focus on topic (a), by discussing the Green's function of a bounded domain Ω. By defn, it is a fn

$$G(x,y) \quad \text{defined for all } x,y \in \Omega \text{ st } x \neq y$$

with the property that

$$-\Delta_y G(x,y) = \delta_x$$
$$G(x,y) = 0 \quad \text{for } y \in \partial\Omega$$

(we think of x being a parameter, + $G(x,y) = u(y)$ the soln of the pde $-\Delta u = \delta_x$ in Ω, $u=0$ at $\partial\Omega$). We'll show shortly that $G(x,y) = G(y,x)$, so while the defn is asymmetric G is actually symmetric.

Utility of this: given G, soln of

$$(*) \quad \begin{aligned} -\Delta u &= f \text{ in } \Omega \\ u &= \varphi \text{ at } \partial\Omega \end{aligned}$$

is
$$u(x) = \int_\Omega G(x,y) f(y)\, dy - \int_{\partial\Omega} \varphi(y)\, \nabla_y G(x,y) \cdot \vec{n}$$

<u>Pf:</u> In general

$$\int_{\Omega} v\Delta u - u\Delta v = \int_{\partial\Omega} v\frac{\partial u}{\partial n} - u\frac{\partial v}{\partial n}$$

Fixing x, take $v(y) = G(x,y)$ + let u solve $(*)$; Then

$$\int_{\Omega} u\Delta v = -\int_{\Omega} u\,\delta_x = -u(x)$$

$$\int_{\Omega} v\Delta u = -\int G(x,y)\,f(y)\,dy$$

$$\int_{\partial\Omega} u\frac{\partial v}{\partial n} = \int_{\partial\Omega} \varphi\ \nabla_y G(x,y)\cdot \vec{n}$$

$$\int_{\partial\Omega} v\frac{\partial u}{\partial n} = 0 \qquad \text{since } v=0 \text{ at } \partial\Omega$$

Does such a G exist? Yes, for any "reasonable" domain. We can "construct" it by taking

$$G(x,y) = \underset{\text{fund soln}}{\Phi(x-y)} + \underset{\text{soln of } \Delta_y v^{(x)}=0 \text{ in } \Omega \text{ with bdry data } v^{(x)} = -\Phi(x-y),\ y\in\partial\Omega.}{v^{(x)}(y)}$$

(provided we accept that solns of $\Delta_y v = 0$ exist

& are nice when the bdry data are nice).

For some special cases G can be made explicit, using tricks like those we used to find the Green's fn for the heat eqn (same name and analogous, but different defn of course!)

<u>For a half-space</u>: if $x = (x', x_n) \in \mathbb{R}^n$ and $\Omega = \{x_n > 0\}$ then

$$G(x,y) = \Phi(y-x) - \Phi(y-\tilde{x})$$

where $\tilde{x}$ = "reflection of x abt $\partial\Omega$" $= (x', -x_n)$. Defining properties of G are clear by inspection.

Note resemblance to what we did for heat eqn in a halfspace. In special case when bc $g = 0$, soln of $-\Delta u = f$ in $\{x_n > 0\}$ with $u = 0$ is obtained by taking odd extension $\tilde{f}$ of f, then solving $-\Delta \tilde{u} = \tilde{f}$ in $\mathbb{R}^n$, then restricting to $x_n > 0$. Since $\tilde{u}$ is odd, it is the desired soln + it is equal to $\int_{x_n>0} G(x,y) f(y)$ with G as given above.

<u>For a ball</u>: similar idea, but "reflection" is

replaced by inversion; for $\Omega = B_1(0)$,

$$G(x,y) = \Phi(y-x) - \Phi(|x|(y-\tilde{x}))$$

where $\tilde{x} = x/|x|^2$. (Crucial fact that makes this work: when $x \in B_1(0)$ & $y \in \partial B_1(0)$, $|y-x| = |x||y-\tilde{x}|$.)

In 2D, focusing on $\Delta u = 0$ with $u = g$ at ∂B, we recover the 2D Poisson Kernel discussed in Lecture 6 notes. In higher dims we get

$$\Delta u = 0 \text{ in } B_r(0) \subset \mathbb{R}^n, \quad u = g \text{ at bdry}$$

$$\Rightarrow \quad u(x) = \frac{r^2 - |x|^2}{n \alpha_n r} \int_{\partial B_r(0)} \frac{g(y)}{|x-y|^n} \, dA_y$$

which is Poisson's kernel for the ball in $\mathbb{R}^n$.

I promised a pf that $G(x,y) = G(y,x)$. Here is a "formal" pf: recall

$$\int_\Omega u \Delta v - v \Delta u = \int_{\partial\Omega} u \frac{\partial v}{\partial n} - v \frac{\partial u}{\partial n}$$

Take $u(y) = G(P,y)$ & $v(y) = G(Q,y)$ with $P \neq Q$.

Then $-\Delta u = \delta_P$, $-\Delta v = \delta_Q$ and both vanish at $\partial\Omega$, So

$$-\int_\Omega u\,\delta_Q + \int_\Omega v\,\delta_P = 0$$

ie

$$G(P,Q) = G(Q,P) .$$

(This pf is easily made honest, by the same argument we used to see that Φ was the fund soln.)

Similar technique can be used to solve Neumann bvp's. Just use $N(x,y)$ defined by

$$-\Delta_y N(x,y) = \delta_x \quad \text{for } y \in \Omega$$

$$\frac{\partial N}{\partial \nu_y} = \text{const} \quad \text{for } y \in \Omega$$

(the value of the constant in the bc is determined by $\int_{\partial\Omega} \partial N/\partial\nu_y = \int_\Omega \Delta_y N = -1$)

N is called the "Neumann function of Ω". Recall that

$$-\Delta u = f \text{ in } \Omega$$
$$\partial u/\partial n = g \text{ at } \partial\Omega$$

has a soln only if data are consistent:

$$\int_{\partial\Omega} g + \int_{\Omega} f = 0,$$

and that (for consistent data) soln is unique only up to an additive constant. We can choose the constant s.t. $\int_{\partial\Omega} u = 0$.

<u>Claim</u>: For the soln normalized this way,

$$u(x) = \int_{\Omega} N(x,y)\, f(y)\, dy + \int_{\partial\Omega} N(x,y)\, g(y)\, dA_y$$

<u>Pf</u>:
$$\int_{\Omega} u\Delta N - N\Delta u = \int_{\partial\Omega} u\frac{\partial N}{\partial \nu} - N\frac{\partial u}{\partial \nu}$$

$$\Rightarrow \quad -u(x) + \int_{\Omega} N(x,y) f(y)\, dy = 0 - \int_{\partial\Omega} N(x,y) g(y)\, dA$$

As for the Green's fn, N is symmetric in $x+y$ (same pf as for G) + we can easily write $N(x,y)$ as sum of $\Phi(x-y)$ and a harmonic correction.