PDE - Lecture 6, 10/7/2014

Note: no class 10/14 (fall break); midterm 10/21.
Also: no office hrs 10/13; replaced by substitute office hr Wed 10/15, 4-5pm.

We began "Laplace-type" eqns last wk. As for heat eqn, our goals are a mixture of

- qualitative properties (mean value principle, smoothness, uniqueness, possible behaviors at ∞)
- explicit soln formulas (by separation of vars, via fundamental soln, a ball, a half-space)
- some basic numerical schemes

Let's turn now to a representation formula for solns of

$$\Delta u = f \quad \text{in all } \mathbb{R}^n$$

subject to condn as $|x| \to \infty$ that assures uniqueness, eg

$$u \to 0 \quad \text{as} \quad |x| \to \infty$$ (reasonable if $n \geq 3$ and f has cpt spt)

or

$$\frac{|u(x)|}{|x|} \to 0 \quad \text{as} \quad |x| \to \infty$$ (more natural if $n=2$, as we'll see very soon)

The main point: in $\mathbb{R}^n$, $n \geq 2$,

$$(*) \qquad u(x) = \int_{\mathbb{R}^n} \Phi(x-y) f(y)\, dy$$

solves $-\Delta u = f$ when

$$(**) \qquad \Phi(z) = \begin{cases} -\frac{1}{2\pi} \log|z| & \text{in } \mathbb{R}^2 \\ c_n |z|^{2-n} & \text{in } \mathbb{R}^n,\ n \geq 3 \end{cases}$$

with the right choice of c_n, namely

$$c_n = \frac{1}{n(n-2)\alpha_n} \qquad \alpha_n = \text{vol of unit ball in } \mathbb{R}^n$$

We'll prove this assuming f is C^2 (though it's true in greater generality). Φ is called the "fundamental solution" of the Laplacian.

What's so special abt the Φ defined by (**)? It's radial, smooth away from $z=0$, and $\Delta_z \Phi = 0$ away from $z=0$. Easy to check the last stmt: $\Delta \Phi = \Phi_{rr} + \frac{n-1}{r}\Phi_r$ for radial fns, so

$$\Delta \Phi = 0 \Rightarrow \Phi_{rr} = -\frac{n-1}{r}\Phi_r$$

$$\Rightarrow (\log \Phi_r)_r = \frac{\Phi_{rr}}{\Phi_r} = \frac{1-n}{r}$$

$$\Rightarrow \log \Phi_r = c_1 + (1-n)\log r$$

$$\Rightarrow \Phi_r = c_2 r^{1-n}.$$

$$\Rightarrow \Phi = \begin{cases} a + b \log r & n = 2 \\ a + \dfrac{b}{r^{n-2}} & n \geq 3 \end{cases}$$

The additive constant isn't interesting; the multiplicative one is a normalization.

OK, let's consider $u(x) = \int \Phi(x-y) f(y)\, dy$. Does $\Delta u = f$? First thought: differentiate under integral. This works OK for $\partial u/\partial x_i$:

$$\frac{\partial u}{\partial x_i} = \int \frac{\partial}{\partial x_i}\Phi(x-y)\, f(y)\, dy$$

since $|\nabla \Phi|$ is integrable (eg in $\mathbb{R}^2$, $\frac{1}{r}$ is integrable near 0).

__But__: calculating $\frac{\partial^2 u}{\partial x_i \partial x_j}$ this way is __wrong__!

In fact, $\nabla\nabla\Phi$ is not integrable at 0, so there's no reason we should be able to differentiate under the integral. (There *is* a formula of the form

$$\frac{\partial^2 u}{\partial x_i \partial x_j} = \int_{PV} \frac{\partial^2}{\partial x_i \partial x_j} \Phi(x-y)\, f(y)\, dy + c_{ij} f(x)$$

where the integral is a "Cauchy principal value", but discussing this would take us too far afield.)

__So__: let's be more careful.

__Step 1__: If f is C^2 and $u(x) = \int \Phi(x-y) f(y)\, dy$ then u is C^2.

__Pf__: let $z = x - y$ so $y = x - z$. Then

$$u(x) = \int \Phi(z) f(x-z)\, dz.$$

__Now__ we can differentiate under the integral.

__Step 2__: Writing

$$\Delta u = \int_{|z|<\varepsilon} \Phi(z)\, \Delta_x f(x-z)\, dz + \int_{|z|>\varepsilon} \Phi(z)\, \Delta_x f(x-z)\, dz$$

we easily see that 1^{st} term $\to 0$ as $\varepsilon \to 0$ (using character of Φ at 0, and hypothesis that f is C^2)

Focusing on 2^{nd} term: since $\Delta_x f(x-z) = \Delta_z f(x-z)$ we can integrate by parts

$$\int_{|z|>\varepsilon} \Phi(z)\, \Delta_z f(x-z)\, dz = -\int_{|z|>\varepsilon} \langle \nabla\Phi, \nabla_z f(x-z) \rangle + \int_{|z|=\varepsilon} \Phi \frac{\partial f}{\partial \nu}$$

and the bdry term $\to 0$ as $\varepsilon \to 0$ (it is of order $\varepsilon \log \varepsilon$ in 2D + $\varepsilon^{2-n} \cdot \varepsilon^{n-1} = \varepsilon$ in $\mathbb{R}^n$, $n \geq 3$)

So: The crucial term is

$$-\int_{|z|>\varepsilon} \langle \nabla\Phi, \nabla_z f(x-z) \rangle = \int_{|z|=\varepsilon} \frac{\partial \Phi}{\partial r} f(x-z)$$

since $\Delta\Phi = 0$ for $|z|>\varepsilon$. (I used $\int_{\Omega} \operatorname{div} \sigma = \int_{\partial\Omega} \sigma \cdot \nu$

with $\Omega = \{|z| > \varepsilon\}$ and $\sigma = f(x-z)\nabla\Phi(z)$.)

Now, $\left.\frac{\partial\Phi}{\partial r}\right|_{r=\varepsilon} = \frac{\text{const}}{\varepsilon^{n-1}}$, and $f(x-z) \approx f(x)$ by conty of f. So as $\varepsilon \to 0$ the limit exists and is a constant times $f(x)$.

The constant in the fund soln is chosen so we get exactly f (we'll see its value more easily in a moment).

Common notation: $\Delta\Phi = \delta_0$, where RHS is a "delta function" i.e. a pt mass at 0. Explain this: recall from step 1 above that

$$\Delta u(x) = \int \Phi(z)\,\Delta_z f(x-z)\,dz$$

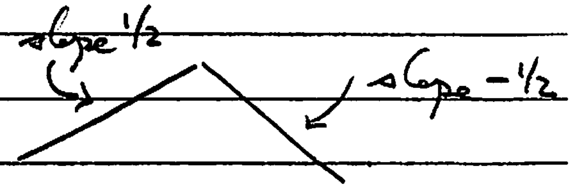
$$= \int \langle -\nabla_z \Phi, \nabla_z f(x-z) \rangle\,dz \quad \underline{\text{honestly}}$$

$$= \int \Delta\Phi \cdot f(x-z)\,dz \quad \underline{\text{formally}}$$

In truth $\Delta u(x) = f(x)$ (we proved this), so the formal calculation is $\underline{\text{true}}$ if we interpret $\Delta_z \Phi = \delta_0$ (acting as a functional on continuous fns)

To gain intuition: what about 1D? There

$-\Delta\Phi = -\Phi_{zz} = \delta_0$ when



since Φ_z is then piecewise constant with a jump of -1 at $z=0$.

One can show, using nothing more than basic Calculus, that $\int_{-\infty}^{+\infty} \Phi(z) f''(z)\, dz = f(0)$ for all nice, compactly-supported f (integrate by parts separately on $(-\infty, 0)$ and $(0,\infty)$).

Situation in $\mathbb{R}^n$ ($n \geq 2$) is a little different since fund soln in $\mathbb{R}^n$ is singular at 0 (that's why we needed to consider $|z| > \varepsilon$ and take a limit as $\varepsilon \to 0$).

An alternative viewpt in $\mathbb{R}^n$ is to regularize the fund soln, considering eg

$$\Phi_\varepsilon(z) = \frac{c_n}{(|z|^2+\varepsilon^2)^{\frac{1-n}{2}}} \quad \text{in } \mathbb{R}^n,\ n \geq 3$$

Then $\Delta\Phi_\varepsilon$ makes perfect sense for $\varepsilon > 0$ and we can study it in the limit $\varepsilon \to 0$. See eg Folland for this approach.

Is there an easy way to get the constant right? Yes indeed. Recall that

$$\int_{\partial B_r(0)} \frac{\partial \Phi}{\partial r} = \int_{B_r(0)} \Delta \Phi$$

Since $-\Delta \Phi = \delta_0$ we expect

$$\int_{\partial B_r(0)} \frac{\partial \Phi}{\partial r} = -1$$

For example, if $\Phi = c_n r^{2-n}$ in $\mathbb{R}^n$ $(n \geq 3)$ then we have

$$c_n \left(\frac{d}{dr} r^{2-n}\right) \cdot r^{n-1} \cdot \left(\begin{array}{c}\text{area of unit} \\ \underline{\text{sphere}} \text{ in } \mathbb{R}^n\end{array}\right) = -1.$$

so

$$c_n (2-n)\, n \alpha_n = -1 .$$

(Similar calc can be done in $\mathbb{R}^n$).

Note that if f has cpt spt then $u(x) = \int \Phi(x-y) f(y)$ has sublinear growth at ∞. So (by the Liouville than we proved using MVP.) This is the <u>unique</u> soln of $-\Delta u = f$ with sublinear growth at ∞.

Fund soln is useful for more than just "knowing the solution". One consequence is <u>another proof that harmonic fns are C^∞</u>:

if $\Delta u = 0$ in a nbh of x_0, then u is C^∞ near x_0.

<u>Pf</u>: consider $\tilde{u} = u\varphi$ where

$\varphi \equiv 1$ near x_0
$\varphi \equiv 0$ where $\Delta u \neq 0$ (or where u is not defined)

Then $\Delta \tilde{u} = \underbrace{2\nabla u \cdot \nabla \varphi + u\Delta\varphi}_{\text{call this } f}$ in all $\mathbb{R}^n$

and $\tilde{u}$ has cpt spt. So

$$\tilde{u}(x) = \int \Phi(x-y) f(y)\, dy$$

Since $f \equiv 0$ near x_0, we can differentiate under the integral ($x \mapsto \Phi(x-y)$ is a smooth function of x near x_0, for every y in $\mathrm{spt}(f)$).

Remaining topics in this segment

a) explicit solns in other settings, by sep of vars or reflection

b) the Green's function for a bounded domain (and its analogue for a Neumann bc)

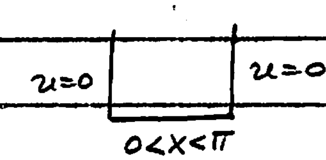
c) The max principle + other qualitative properties

d) basic numerical schemes (finite differences and variational methods)

The separation of variables part of (a):

Prob Set 5 has lots of sep of vars in the plane. But there are other settings where this technique gives a lot of insight as well:

① a half-strip in $\mathbb{R}^2$



$\Rightarrow$ take "discrete Fourier transform in x"

i.e. look for $u(x,y) = \sum a_k(y) \sin kx$

Easy to see that $\Delta u = 0 \Rightarrow$

$$a_k(y) = c_k e^{ky} + d_k e^{-ky}$$

Need a growth restriction at $y \to \infty$ to make soln uniquely determined by data at $0 < x < \pi$, $y = 0$. For ex: u bounded as $y \to \infty$ rules out the e^{ky} terms

(2) a halfspace, eg $\left.\begin{array}{l} \Delta u = 0 \text{ for } y > 0 \\ u = u_0 \text{ at } y = 0 \end{array}\right\}$ in $\mathbb{R}^2_+$.

Almost the same as (1), except we must now use the (cont's) Fourier transform wrto x:

$$u(x,y) = c \int_{\mathbb{R}^{n-1}} \hat{u}_0(\xi)\, e^{-|\xi| y}\, e^{i\xi \cdot x}$$

for $\mathbb{R}^n_+ = \{(x,y) : x \in \mathbb{R}^{n-1}, y > 0\}$,

(3) periodic bdry conds, eg
$\Delta u = f$ where u + f are periodic in each variable with period 2π

Note consistency cond in this case: $\displaystyle\int_{\text{period cell}} f = 0$

is req'd for existence (since $\int_{cell} \Delta u = \int_{\partial cell} \partial u/\partial \nu = 0$ due to periodicity).

Very much like ①, but now use discrete Fourier transform in all vars: using complex notation

$$f(x) = \sum_{k \in \mathbb{Z}^D}{}' \hat{f}(k)\, e^{ik\cdot x}$$

Then $\Delta u = f \iff -|k|^2 \hat{u}(k) = \hat{f}(k) \qquad k \neq 0$.
(Consistency cond is evident: LHS $=0$ when $k=0$).
Note: soln is unique only up to an additive constant, since $\hat{u}(0)$ is arbitrary.

In periodic setting it's easy to see that

$$f \text{ has } k \text{ derivs in } L^2 \implies \frac{\partial^2 u}{\partial x_i \partial x_j} \text{ has } k \text{ derivs in } L^2 \text{ for each } i,j$$

(Use Plancherel's formula)

Next, let's do the finite-difference part of (d) (for a concise discn of this topic see W. Strauss's book).

1D first: $x_j = \frac{j}{N}$ $\quad x_0 = 0,\ x_N = 1$

Finite difference version of $u_{xx} = f$ on $[0,1]$, $u = 0$ at endpts is

$$\frac{u_{j+1} + u_{j-1} - 2u_j}{(\Delta x)^2} = f_j \qquad j = 1, 2, \ldots, N-1,$$

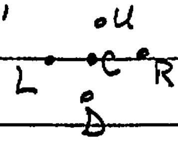
with convention $u_0 = u_N = 0$. This can be expressed as $K\vec{u} = \vec{f}$

$$\frac{1}{(\Delta x)^2}\begin{bmatrix} -2 & 1 & & & & \\ 1 & -2 & 1 & & 0 & \\ & 1 & -2 & 1 & & \\ & & & \ddots & & \\ & 0 & & 1 & -2 & 1 \\ & & & & 1 & -2 \end{bmatrix}\begin{bmatrix} u_1 \\ \vdots \\ \\ u_{N-1} \end{bmatrix} = \begin{bmatrix} f_1 \\ \vdots \\ \\ f_{N-1} \end{bmatrix}.$$

Matrix is tridiagonal (and very well understood). Of course it's invertible; in fact same algebra gives

$$\langle K\vec{u}, \vec{u}\rangle = -\sum_{j=0}^{N-1}\left(\frac{u_j - u_{j+1}}{\Delta x}\right)^2$$

(always using convention $u_0 = 0$, $u_N = 0$). This is the discrete version of $\int u\Delta u = -\int|\nabla u|^2$ when $u = 0$ at bdry.

In 2D (in a rectangle) the situation is very similar, but now the discrete Laplacian has a "5 pt stencil" 

$$\Delta u(C) = \frac{u_R + u_L + u_U + u_D - 4u_C}{(\Delta x)^2}$$

and the matrix is sparse but no longer tridiagonal. (There is, however, still a discrete analogue of integn by parts, as in 1D.)

Intuition gained from this: solving a Laplace-type problem in a bounded domain is a lot like solving a linear system (in fact, one with rather special structure).

Let's turn now to the max principle.

wk form of max prin says: $\Delta u = 0$ in Ω, with $\Omega \subset \mathbb{R}^n$ bounded $\Rightarrow$ u achieves its max + min values at $\partial\Omega$.

Pf can be done along same line we used for heat eqn (when $\Delta u < 0$ or $\Delta u > 0$ conclusion follows by elementary calculus; general case is obtained by considering $u_\varepsilon(x) = u(x) \pm \varepsilon |x|^2$ and taking $\varepsilon \to 0$).

Lots of extensions + applns (situation is a lot like what we did for heat eqn).

<u>But</u>: for $\Delta u = 0$ we can use MVP to prove

a stronger result –

<u>strong form of max prin</u>: if max or min of u is achieved at an interior pt of a bounded, connected set Ω (where $\Delta u = 0$ in Ω) then u must be <u>constant</u>.

<u>Pf</u>: Suppose $\max_{x \in \Omega} u(x) = M$, and consider

$$S = \{ x \in \Omega : \quad u(x) = M \}.$$

It is closed, since (by hypothesis) u is conts. But it's also <u>open</u>, by MVP:

$$\text{if } u \le M \text{ in } B_r \text{ and } u(x) = \fint_{B_r} u = M$$

then we easily conclude that $u \equiv M$ in B_r.

So: if Ω is connected, S is either empty or else all Ω.

(Analogous results are true in parabolic setting and for variable-coefft eqns, but their proof lies beyond this class.)

Here's something that belongs logically with the "separation of variables" discussion done earlier:

If $\Delta u = 0$ in the 2D ball $x_1^2 + x_2^2 < a^2$ and $u = f$ (viewed as a function of angle) at the bdry then

$$u(r,\theta) = \frac{1}{2\pi}\int_0^{2\pi} \frac{a^2 - r^2}{a^2 + r^2 - 2ar\cos(\varphi - \theta)} f(\varphi)\, d\varphi$$

(in polar coords), or equivalently

$$u(x) = \frac{a^2 - |x|^2}{2\pi a} \int_{\partial B(0,a)} \frac{f(y)}{|x-y|^2}\, dA_y .$$

This is the 2D version of Poisson's formula (there is also an n-dim'l version, see eg Evans Section 2.2). In 2D it follows easily from the separation of variables repn discussed at length in Prob Set 5, by directly summing the series (see eg Guenther + Lee §8-2 or else Strauss for this calculation).

An immediate consequence: whereas the MVP only gives us $u(0) =$ avg of bdry data, the Poisson formula tells us

- $u(x) =$ weighted average of bdry data (different weights at each x)

- another pf that $\Delta u = 0$ in a ball $\Rightarrow u$ is C^∞ in the ball (differentiate under the integral, the Euclidean version is better for this)

- convenient estimates for ∇u (or even higher derivs of u) in terms of the bdry data.