PDE - Lecture 5, 9/30/2014

We turn now to Laplace's eqn, and related problems.

First, some motivating examples

(1) Recall that if φ + f are indep of t, then soln of

$$u_t - \Delta u = f \quad \text{in } \Omega \quad \text{for } t>0$$
$$u = \varphi \quad \text{at } \partial\Omega$$

(Ω = bounded domain in $\mathbb{R}^n$) approaches the steady-state u_* as $t \to \infty$, where

$$-\Delta u_* = f \quad \text{in } \Omega$$
$$u_* = \varphi \quad \text{at } \partial\Omega$$

Thus, in studying u_* we are effectively considering the steady-state behavior of various evolutions considered in Lecture 1 as motivations for the heat eqn (eg heat transfer).

The same applies in all $\mathbb{R}^n$, if eg $u(x,t) \to 0$ as $|x| \to \infty$. For example we expect soln to $u_t - \Delta u = f$ in $\mathbb{R}^n$ (with f indep of t) to decay to soln of $-\Delta u_* = f$ in $\mathbb{R}^n$ (with $u \to 0$ as $|x| \to \infty$)

as $t \to \infty$. (The condn at ∞ plays the role of a boundary condition in this case.)

Similarly: solns of $-\Delta u = f(u)$ arise in considering steady-states of population dynamics laws.

Still similar: we saw in HW1 that a biased random walk provides a finite-difference analogue of $u_t = u_{xx} - (\alpha(x)u)_x$. As $t \to \infty$ we may expect convergence to a soln of $u_{xx} - (\alpha(x)u)_x = 0$ representing a steady-state prob distribution.
But: is the steady-state distrn nonzero? For $\alpha = 0$ the answer is no (if $u_t = u_{xx}$ with $u \to 0$ as $|x| \to \infty$ then $u \to 0$ as $t \to \infty$). But for other choices of $\alpha(x)$ the answer may be yes. For example if the bias always pushes the walker toward 0, say $\alpha(x) = -x$, then the steady-state will be $u = c\exp(-\frac{1}{2}x^2)$, which solves $u_{xx} + (xu)_x = 0$. The same discn applies in $\mathbb{R}^n$, for multidimensional (biased) random walks.

(2) Another probabilistic interpretation: expected arrival time at bdry

1D version: for a random walker on a grid, moving left or right with prob 1/2 and time step

Δt st $(\Delta x)^2/2\Delta t = 1$, if

$u(j\Delta x) = u_j$ = expected time when walker reaches bdry, starting from $j\Delta x$ at time 0

we have

$$u_j = \underbrace{\tfrac{1}{2} u_{j-1} + \tfrac{1}{2} u_{j+1}}_{\substack{\text{expected} \\ \text{arrival time} \\ \text{starting from} \\ \text{next time step}}} + \Delta t$$

$$\Rightarrow \quad u_{j-1} + u_{j+1} - 2u_j = -2\Delta t = -(\Delta x)^2.$$

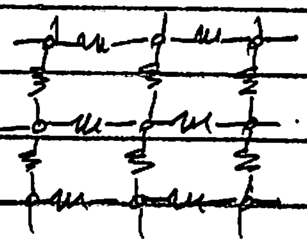
Discretization of $u_{xx} = -1$. Bdry cond is $u = 0$ at $\partial\Omega$.

In 2D, if walker goes to each of 4 nearest neighbors with equal prob, get similarly a discretization of

$$u_{xx} + u_{yy} = -1 \quad \text{in } \Omega$$
$$u = 0 \quad \text{at } \partial\Omega$$

provided $\dfrac{4\Delta t}{(\Delta x)^2} = 1$.

(3) <u>Electrostatics</u> — can be done in a continuum setting but let's consider a 2D resistor network



spatial mesh h.

u_{ij} = voltage at i,jth node (point (ih, jh))

current flow from (i,j) to $((i+1),j)$ is proportional to $u_{i+1,j} - u_{i,j}$ (Ohm's law). Same for vertical links.

Total current balance at node (i,j):

$$u_{i+1,j} + u_{i-1,j} + u_{i,j+1} + u_{i,j-1} - 4u_{ij} = 0$$

→ finite difference approxn of $u_{xx} + u_{yy} = 0$.

(4) <u>Complex Variables</u>: $f(z)$ analytic ⇒ $f = u + iv$ with u + v harmonic fns of $x+y$, where $z = x + iy$

(5) <u>Incompressible, irrotational flow</u> (see eg Kevorkian, opening of Chap 2)

$\bar{q}$ = fluid velocity (a vector field)

$$\operatorname{div} \vec{q} = 0 \iff \text{incompressibility}$$
$$\operatorname{curl} \underset{\sim}{q} = 0 \iff \text{flow is "irrotational"}$$

The latter implies $\underset{\sim}{q} = \nabla \varphi$, whence $\operatorname{div} \underset{\sim}{q} = 0 \Rightarrow \Delta \varphi = 0$

(6) Variational problems: if u_* achieves

$$\min_{u=\varphi \text{ at } \partial\Omega} \int_\Omega W(\nabla u)\, dx$$

Then for any v st $v|_{\partial\Omega} = 0$ we have

$$t \to \int_\Omega W(\nabla u_* + t\nabla v) \quad \text{has a min at } t=0$$

so, by 1^{st} deriv test,

$$\int_\Omega \operatorname{div}\left(\frac{\partial W}{\partial \nabla u}(\nabla u_*)\right) \cdot v \, dx = 0$$

True for all v $\Rightarrow$

$$\operatorname{div}\left(\frac{\partial W}{\partial \nabla u}(\nabla u_*)\right) = 0$$

(This process is called "considering the 1^{st} variation". We actually did it before, when we identified the gradient of $\int_\Omega W(\nabla u)\,dx = E[u]$.)

Special cases:

- $$\min_{\substack{u=\varphi \\ \text{at } \partial\Omega}} \int_\Omega |\nabla u|^2 \, dx \iff \begin{array}{l} \Delta u = 0 \text{ in } \Omega \\ u = \varphi \text{ at } \partial\Omega \end{array}$$

- $$\min_{(\text{no bc})} \int_\Omega \tfrac{1}{2}|\nabla u|^2 + f u \, dx \iff \begin{array}{l} -\Delta u + f = 0 \text{ in } \Omega \\ \partial u/\partial n = 0 \text{ at } \partial\Omega \end{array}$$

(for the latter: no bdry condn on u $\Rightarrow$ no restrn on v at $\partial\Omega$ $\Rightarrow$ if u_* achieves min then

$$\int_\Omega \nabla u_* \cdot \nabla v + f v \, dx = 0 \quad \text{for all } v$$

$$\Rightarrow \int_\Omega (-\Delta u_* + f) v + \int_{\partial\Omega} \frac{\partial u_*}{\partial n} v = 0 \quad \text{for all } v$$

True for all v with cpt spt $\Rightarrow$ $-\Delta u_* + f = 0$.
Now consider v nonzero on bdry $\Rightarrow$ $\partial u_*/\partial n = 0$.)

- $$\min_{u=\varphi \text{ at } \partial\Omega} \int_\Omega (1+|\nabla u|^2)^{1/2} \quad (\text{area of graph of } u)$$

$$\iff \operatorname{div} \frac{\nabla u}{(1+|\nabla u|^2)^{1/2}} = 0 \quad (\text{"minimal surface eqn"})$$

$$u = \varphi \text{ at } \partial\Omega$$

Turning now to some theory (the next couple of lectures correlate strongly with Evans Chap 2.2 + Guenther and Lee Chapter 8), let's start with some easy observations:

1st Recall that lin heat eqn was like an infinite-dimensional (steepest-descent) ODE. Does the bvp

$$\Delta u = f \text{ in } \Omega$$
$$u = \varphi \text{ at } \partial\Omega$$

have a similarly familiar analogy?

Ans: yes, it's like an (invertible) system of linear eqns. For example, if Ω = rectangle in 2D + we use the finite-difference discretization

$$\frac{1}{h^2}\left(u_{i,j+1} + u_{i,j-1} + u_{i+1,j} + u_{i-1,j} - 4u_{ij}\right) = f_{ij}$$

then the bdry nodes are fixed by the bdry data, + we have a system of linear eqns for the u_{ij}'s assoc to the interior nodes.

2nd In a bdd domain, $\Delta u = f$ in Ω
$u = \varphi$ at $\partial\Omega$

can have at most one soln (in fact, there is a soln, obtained eg by solving $\min_{u=\varphi \text{ at } \partial\Omega} \int_\Omega \left(\frac{1}{2}|\nabla u|^2 + fu\right)$)

Pf of uniqueness: subtract 2 solns $\Rightarrow$ diff solves $\Delta u = 0$ in Ω, $u = 0$ at $\partial\Omega$. Multiply by u + integrate by pts to see

$$\int_\Omega u\Delta u = 0 \Rightarrow \int_\Omega |\nabla u|^2 = 0 \Rightarrow u = \text{const}$$

Now bdry cond $\Rightarrow u = 0$.

[Uniqueness also follows from max princ, to be discussed a little later]

3d Eqn $\Delta u = f$ in Ω
$\frac{\partial u}{\partial n} = \psi$ at $\partial\Omega$

can have a soln only if ψ + f satisfy the consistency cond $\int_{\partial\Omega} \psi \, dA = \int_\Omega f \, dx$

For consistent data, soln is unique up to an additive constant (in fact it exists; we'll discuss this later)

Pf of consistency condn: $\int_\Omega \Delta u = \int_{\partial\Omega} \partial u/\partial n$ for <u>any</u> u, by Gauss' Thm.

Pf of uniqueness up to const: proceed exactly as before, but now we must stop at

$$\int_\Omega |\nabla u|^2 = 0 \Rightarrow u = \text{const}.$$

4th <u>Why</u> is there a consistency condn for the Neumann problem?

Recall that Δ is self-adjoint for the L^2 inner product, with either the bc $u=0$ at $\partial\Omega$ or the bc $\partial u/\partial n = 0$ at $\partial\Omega$.

Also recall that in finite-dimensional lin algebra, if A is symmetric $n \times n$ matrix then $Az = b$ has a soln iff $b \perp \ker A$.

So: Neumann pbm has a consistency cond since there's a nontrivial kernel (the constant functions) solving $\Delta u = 0$ when the bc is $\partial u/\partial n = 0$ at $\partial\Omega$.

Turning to something deeper, let's discuss the Mean Value Property ("MVP") for harmonic fns. I start with this since

a) it's elegant + elementary
b) it has lots of important consequences

Mean value property: if u is C^2 and $\Delta u = 0$ then

$$u(x) = \fint_{\partial B(x,r)} u \, dA = \fint_{B(x,r)} u \, d\mathrm{vol}$$

where the slashed integrals denote averages (eg in $\mathbb{R}^2$, $\fint_{\partial B(0,r)} f = \frac{1}{2\pi}\int_0^{2\pi} f(re^{i\theta})\, d\theta$,

$$\fint_{B(0,r)} f = \frac{1}{\pi r^2}\int_0^r \int_0^{2\pi} f(\rho e^{i\theta})\, \rho \, d\rho \, d\theta \quad)$$

Pf: Let $\varphi(r) = \fint_{\partial B(x,r)} u = \frac{1}{|\partial B(0,1)|}\int_{\partial B(0,1)} u(x+ry)\, dA_y$

Evidently

$$\varphi'(r) = \fint_{\partial B(x,r)} \frac{\partial u}{\partial n}\, dA$$

But $\int_{\partial B(x,r)} \partial u/\partial n \, dA = \int_{B(x,r)} \Delta u \, dx = 0$. So

φ is indep of r. But clearly (since u is conts)

$$\varphi(r) \to u(x) \quad \text{as} \quad r \to 0.$$

So $\varphi(r) = u(x)$ for all x, provided only that u is harmonic in $B_r(x)$. This proves 1st part of MVP

$$\fint_{\partial B(x,r)} u = u(x) \qquad \text{for all } r$$

The 2nd assertion follows easily by integn wrt r ("method of shells")

$$\int_{B(x,r)} u \, dx = \int_0^r \left[\int_{\partial B(x,\rho)} u \, dA \right] d\rho$$

$$= u(x) \cdot \int_0^r \left[\text{Area of } \partial B(x,\rho) \right] d\rho$$

$$= u(x) \cdot |B(x,r)| \, .$$

MVP has many consequences – we'll return to collect some of them later – but here is one of the most important ones:

<u>Liouville's Thm for harmonic fns on $\mathbb{R}^n$</u>: if $\Delta u = 0$ on $\mathbb{R}^n$ + u is unif bounded then $u \equiv$ const.

Proof: Observe (assuming u is C^3) that

$$\Delta u = 0 \Rightarrow \Delta \frac{\partial u}{\partial x_i} = 0$$

Now

$$\frac{\partial u}{\partial x_i}(x_0) = \fint_{B(x_0,r)} \frac{\partial u}{\partial x_i}\, dx \qquad \text{by MVP}$$

$$= \frac{c}{r^n} \int_{\partial B(x_0,r)} u \cdot n_i \, dA$$

using, for the last step, that $\operatorname{div}(u e_i) = \langle \nabla u, e_i \rangle = \partial u / \partial x_i$ when $e_i = i^{th}$ basis vector of $\mathbb{R}^n$.

If u is unif bdd then we conclude

$$\left| \frac{\partial u}{\partial x_i}(x_0) \right| \leq \frac{\text{const}}{r} \cdot \max |u| ,$$

whence as $r \to \infty$ we see that $\nabla u \equiv 0$.

Rmk: We actually proved a bit more: if

$$\frac{|u(x)|}{|x|} \to 0 \quad \text{as } |x| \to \infty$$

then u is constant. Thus: a nonconstant harmonic fn on all $\mathbb{R}^n$ must grow at least linearly at ∞.

(Obviously, a harmonic fn that grows linearly at ∞ need not be constant; for example, any linear fn is harmonic.)

Hypoth above was that u is C^3. But actually, if $\Delta u = 0$ (in any reasonable sense) then u is C^∞. This too comes easily from the MVP: let

$$u_\varepsilon = \text{"}u \text{ convolved with a spherical mollifier"}$$

i.e.

$$u_\varepsilon(x) = \int u(x-y)\, \varphi_\varepsilon(y)\, dy$$

where $\varphi_\varepsilon(y) = \varepsilon^{-n} \Phi(y/\varepsilon)$ with Φ compactly spt'd, radial, $\int \Phi(z)\, dz = 1$. Then φ_ε behaves like a δ-fn as $\varepsilon \to 0$, i.e.

$$u_\varepsilon(x) \to u(x) \quad \text{as } \varepsilon \to 0 \text{ for any cont's fn } u.$$

Moreover u_ε is C^∞ (if Φ is C^∞), since we can differentiate under the integral. But if u satisfies the MVP then $u_\varepsilon(x) = u(x)$ <u>exactly</u> for all $\varepsilon > 0$, since

$$u_\varepsilon(x) = \int dr \left(\int_{|y|=r} u(x-y)\, \varphi_\varepsilon(y)\, dA_y \right)$$

$$= u(x) \int dr \left(\int_{|y|=r} \varphi_\varepsilon(y)\, dA_y \right) = u(x)$$

so u (being equal to u_ε) is C^∞.