Today's topics:

a) heat eqn in all $\mathbb{R}^n$
b) heat eqn in a halfspace
c) nonzero (and nonstationary) initial data

Consider

\[ u_t - \Delta u = 0 \text{ for } x \in \mathbb{R}^n, \ t > 0 \]
\[ u = u_0(x) \text{ at } t = 0 \]

What carries over from our prior discussions?

- In $\mathbb{R}^n$, there are no longer $L^2$ eigenvalues of $\Delta$. But we'll get an excellent solution formula in a moment, using the fundamental solution

- Explicit difference is conceptually OK, but impractical since space is unbounded

- Implicit differencing is fine if $\|u\|^2 dx$ is finite

- Uniqueness via energy method is OK, provided $\|u\|^2 dx$ is finite.
Solution formula: if $u_0(x)$ is compactly supported in $\mathbb{R}^n$ then the solution is

$$u(x,t) = \int_{\mathbb{R}^n} \phi(y-x,t) u_0(y) \, dy$$

with

$$\phi(z,t) = \frac{1}{(4\pi t)^{n/2}} e^{-|z|^2/4t}$$

(\phi is called the "Fundamental Solution").

Sketch of proof:

- $\phi_t - \Delta \phi = 0$ (easy to check directly);
  so $u$ solves $u_t - \Delta u = 0$ for $t > 0$.

- $\int_{\mathbb{R}^n} \phi(z,t) \, dz = 1$ for any $t$
  (easy to check, using that $\int_{-\infty}^{+\infty} e^{-x^2} \, dx = \sqrt{\pi}$)
4.3

- as \( t \to 0 \), \( \Phi(y-x,t) \) is highly concentrated near \( y = x \)

\[ y \to \Phi(y-x,t) \text{ is a Gaussian with mean } x \text{ and variance } t \]

\[ \int \Phi(x-y,t) u_0(y) \, dy = \int_{|y-x|<\varepsilon} + \int_{|y-x|>\varepsilon} \]

As \( t \to 0 \), 1st term \( \to u_0(x) \) using

center of \( u_0 \); 2nd term \( \to 0 \), using

boundedness of \( u_0 \).

Note: 1) Since \( \Phi(z,t) \) decays rapidly as \( |z| \to \infty \),

sole formula does not require that \( u_0 \) decay

at \( \infty \). In fact it can grow rather rapidly.

2) Uniqueness can be proved in a class that

permits rapid growth at \( \infty \), eg assuming only that

\[ u_0(x,t) \leq C e^{ax^2} \]

for some \( C, a > 0 \). (See eg Evans,

Chapter 2, or [for the 1D version] Guenther + Lee,

Thm 4-3)

How could we have guessed the sole formula?

Method 1 & 2: Recall from Lecture 1 that

finite
difference approx of heat eqn is assoc to a rain-flipping random walk. So pde should be assoc to a limit of random walks. Probabilistic
interpretation of soln formula

\[ \Phi(y-x,t) = \text{prob of being at } y \text{ at time } t, \]
for a walker who starts at \( x \) at time 0

By central limit theorem, distribution of positions after many rain flips is asymptotically Gaussian. So \( \Phi(x,t) \) must be a Gaussian w/ params \( t \).

(Dependency on \( t \) is thin, easily deduced from fact that \( \frac{\partial \Phi}{\partial t} - \frac{t}{2} \Phi = 0 \).)

Method 2: Laplacian does have eigendecomposition in \( \mathbb{R}^n \), namely, \( e^{i\xi \cdot x} \) for any \( \xi \in \mathbb{R}^n \). They're not in \( L^2 \), and they form a continuum rather than a countable family. But the analogue of our \( L^2(\mathbb{S}^2) \) basis expansion is the Fourier transform

\[ f(x) = (2\pi)^{-n/2} \int e^{i\xi \cdot x} \hat{f}(\xi) \, d\xi \]

\[ \hat{f}(\xi) = (2\pi)^{-n/2} \int e^{-i\xi \cdot x} f(x) \, dx \]

Since \( \Delta (e^{i\xi \cdot x}) = -|\xi|^2 e^{i\xi \cdot x} \), rely of initial
value phi should be
\[ u(x,t) = (2\pi)^{-n/2} \int e^{-i \frac{x^2 t}{2} + i \frac{y^2}{2}} e^{-i y \cdot x} dy \, d\xi \]
\[ = (2\pi)^{-n/2} \int e^{-i \frac{x^2 t}{2}} e^{-i y \cdot x} dy \, d\xi \]
\[ = \int \Phi(x-y, t; x, y) dy \]

with
\[ \Phi(z, t) = (2\pi)^{-n} \int e^{-i \frac{x^2 t}{2} + i \frac{z^2}{2}} dy \]

Doing the integration: if \( \eta = \sqrt{t} - \frac{i z}{2\sqrt{t}} \) then
\[ \Phi(z, t) = (2\pi)^{-n} \int e^{-i \frac{\eta^2 t}{4} - \eta^2} \, e^{-i \frac{z^2}{2\sqrt{t}}} \, d\eta \]
\[ = (2\pi)^{-n} \int e^{-i \frac{\eta^2 t}{4}} - \eta^2 - \frac{z^2}{2\sqrt{t}} \, d\eta \]
\[ = \pi^{n/2} e^{-\eta^2} \int e^{-i \frac{\eta^2 t}{4}} \frac{d\eta}{\sqrt{t}} \]
\[ = (2\pi t)^{-n/2} e^{-\frac{z^2}{2\sqrt{t}}} \]

as expected.

Some features of this explicit solution formula:

- We easily see that \( u \) is instantly \( C^\infty \)
- We matter how long \( u \) might be, by
differentiating under the integral

- We see that ill-posedness of solving heat eqn backward in time is linked to difficulty of "deblurring" badly focused photos (blens from bad focus is similar to convolution with a Gaussian)

Can our "solve formula" in a bounded domain be put in a similarly explicit form, when the bc is either \( u = 0 \) at \( \partial \Omega \) or \( \partial u / \partial n = 0 \) at \( \partial \Omega \)? Yes indeed! Forming \( u = 0 \) at \( \partial \Omega \) to be specific, we have that

\[
\begin{align*}
  u_t - \lambda u &= 0 \quad \text{in } \Omega \quad \text{(bounded)} \\
  u &= 0 \quad \text{at } \partial \Omega \\
  u &= u_0 \quad \text{at } t = 0
\end{align*}
\]

\[\Rightarrow u(x, t) = \int_{\Omega} G(x, y; t) u_0(y) \, dy\]

with

\[
G(x, y; t) = \sum_{n=1}^{\infty} e^{-\lambda n^2 t} \phi_n(x) \phi_n(y)
\]

This is just a rewrite of our prev solve formula. (Recall \( \phi_n \) are an orthonormal basis of \( L^2(\Omega) \).)
\[-\Delta \psi = \lambda \psi, \text{ with } \psi = 0 \text{ at } \partial D.\] Note that 
\(G\) is symmetric in \(x + y\), but no longer a function of \(x - y\).

\(G(x, y; t)\) is called the "Greens function" of
the heat eqn with bc \(\psi = 0\) at \(\partial D\). (There is
an entirely analogous system with bc \(\partial u/\partial n = 0\)
at \(\partial D\), obtained using superimposing a unit
with that bc).

What about heat eqn in a half-space? I'll focus on the 1D case (a half-line) to keep
the note from simple, but the same ideas apply to \(\mathbb{R}^n \times x > 0\).

For Dir bc \((\psi = 0 \text{ at } \partial D)\) or Neumann bc \((\partial u/\partial n = 0 \text{ at } \partial D)\) there is easier, by
reflection. Key pt is that

\[\text{if } u(x) = u(-x) \text{ then } u(0) = 0\]
(assuming \(u\) is differentiable)

\[\text{if } u(x) = -u(-x) \text{ then } u(0) = 0\]
(assuming \(u\) is continous)

by using either the odd or even
extension of the initial data we get a soln of heat eqn in all $\mathbb{R}^n$

That's correspondingly odd or even in $x$ for all $t > 0$.

Thus:

\[ u_t - u_{xx} = 0 \text{ for } x > 0, \quad u = 0 \text{ at } x = 0 \]

\[ \Rightarrow u(x,t) = \int_0^\infty G(x,y,t) u_0(y) \, dy \]

\[ G(x,y,t) = \Phi(x-y,t) - \Phi(x+y,t) \]

Similarly:

\[ u_t - u_{xx} = 0 \text{ for } x > 0, \quad u = 0 \text{ at } x = 0 \]

\[ \Rightarrow u(x,t) = \int_0^\infty N(x,y,t) u_0(y) \, dy \]

\[ N(x,y,t) = \Phi(x-y,t) + \Phi(x+y,t) \]

What about nonzero body heat? I'll focus on 1D half-line. Case of half-space in $\mathbb{R}^n$ or bounded $\Omega \subset \mathbb{R}^n$ is essentially the same. I'll do Dirichlet, but Neumann BC can be done similarly (this will be HW). So consider:

\[ u_t - u_{xx} = 0 \text{ for } x > 0 \]

\[ u = g(t) \text{ at } x = 0 \]

\[ u = 0 \text{ at } t = 0 \]

\[ u = 0 \text{ for } x > 0 \]}
Claim: solution is
\[ u(x,t) = \int_0^t \int_{\mathbb{R}^d} G(x, y; t-s) q(s) dy ds, \]

where $G$ is the Green's function of the half-space problem with a homogeneous Dirichlet (formula (6) on pg 4.8). After some arithmetic this amounts to
\[ u(x, t) = \int_0^t \int_{\mathbb{R}^d} \frac{t^d}{(4\pi t)^{d/2}} e^{-\frac{x^2}{4t}} q(s) dy ds, \]

Proof: fix $(x_0, t_0)$ and consider
\[ v(y, T) = G(x_0, y; t_0 - T). \]

It solves heat eqn backward in time, with a $\delta$ for specified data at $y = x_0$ as $T \to 0$. So for $0 < s < t_0$,
\[ \frac{\partial}{\partial s} \int_0^\infty v(y, s) u(y, t) dy = \int_0^\infty v_y u + v u_y dy = \int_0^\infty - (\Delta) u + v (\Delta u) dy = - v_y u_{||} + v u_{||}. \]

The limits as $y \to \infty$ vanish (since $v \to 0$ and $y \to 0$ there).
\[ \text{RHS} = v_y u_{||} - v u_{||} \bigg|_{y = 0}^{y = 0} = \frac{\partial}{\partial y} G(x, 0; t_0 - s). \]
But
\[
\lim_{t \to 0^+} \int_{-\infty}^{\infty} v(y, t) u(y, t) \, dy = u(x_0, t_0)
\]

\[
\lim_{t \to 0^+} \int_{-\infty}^{\infty} v(y, t) u(y, t) \, dy = 0 \quad \text{since by hypothesis}
\]
\[
u = 0 \text{ at } t = 0.
\]

Thus by fund theorem of calc
\[
u(x_0, t_0) = \int_{0}^{t_0} \left( \frac{\partial}{\partial t} \int_{-\infty}^{\infty} v(y, t) u(y, t) \, dy \right) \, dt
\]

\[
= \int_{0}^{t_0} G_y(x_0, t-s) g(s) \, ds
\]
as expected.