PDE - Lecture 4, 9/23/2014

Today's topics:

a) heat eqn in all $\mathbb{R}^n$
b) heat eqn in a halfspace
c) nonzero (and nonstationary) bdry data

Consider now $u_t - \Delta u = 0$ for $x \in \mathbb{R}^n$, $t > 0$
$u = u_0(x)$ at $t=0$

What carries over from our prev discussions?

- In $\mathbb{R}^n$, there are no longer L^2 eigenfns of Δ. But we'll get an excellent soln formula in a moment, using the "fundamental soln"

- Finite differences are conceptually OK but impractical since space is unbounded

- Implicit differencing in time is OK, provided data are such that $\int |\nabla u|^2 dx$ is finite

- Uniqueness via energy method is OK, provided $\int |\nabla u|^2 dx$ is finite.

- Uniqueness via max prin is OK, with hypoth on growth at ∞ (we had a HW pblm like this, but better results are possible)

<u>Solution formula</u>: if $u_0(x)$ is cont's and unif bounded on $\mathbb{R}^n$ then soln is

$$u(x,t) = \int_{\mathbb{R}^n} \Phi(y-x,t)\, u_0(y)\, dy$$

with

$$\Phi(z,t) = \frac{1}{(4\pi t)^{n/2}} e^{-|z|^2/4t}$$

(Φ is called the "Fundamental Soln").

<u>Sketch of pf</u>:

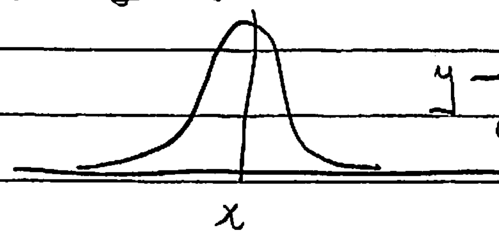
- $\Phi_t - \Delta \Phi = 0$ (easy to check directly);

  so u solves $u_t - \Delta u = 0$ for $t > 0$.

- $\int_{\mathbb{R}^n} \Phi(z,t)\, dz = 1$ for any t

  (easy to check, using that $\int_{-\infty}^{+\infty} e^{-x^2} dx = \sqrt{\pi}$)

- as $t \to 0$, $\Phi(y-x,t)$ is highly concentrated near $y=x$



$y \to \Phi(y-x,t)$ is a Gaussian with mean x + variance $\sim t$

- $$\int \Phi(x-y,t)\, u_0(y)\,dy = \int_{|y-x|<\varepsilon} \text{——} + \int_{|y-x|>\varepsilon} \text{——}.$$

As $t \to 0$, 1st term $\approx u_0(x)$ using cont'y of u_0; 2nd term ≈ 0, using boundedness of u_0.

<u>Note:</u> 1) since $\Phi(z,t)$ decays rapidly as $|z| \to \infty$, soln formula does <u>not</u> require that u_0 decay at ∞. In fact it can grow rather rapidly.

2) Uniqueness can be proved in a class that permits rapid growth at ∞, eg assuming only that $|u(x,t)| \le C e^{a|x|^2}$ for some $C, a > 0$. (See eg Evans Chapter 2, or [for the 1D version] Guenther + Lee Thm 4-3)

<u>How could we have guessed the soln formula?</u>

<u>method 1</u>: Recall from Lecture 1 that finite

difference approxs of heat eqn is assoc to a coin-flipping random walk. So pde should be assoc to a limit of random walks. Probabilistic interpretn of soln formula

$\Phi(y-x,t) =$ prob of being at y at time t, for a walker who starts at x at time 0

By central limit Thm, distn of positions after many coin flips is asymptotically Gaussian, so $\Phi(z,t)$ must be a Gaussian wrt z. (Dependence on t is then easily deduced from fact that $\partial_t \Phi - \Delta_z \Phi = 0$.)

Method 2: Laplacian does have eigenfunctions in $\mathbb{R}^n$, namely $e^{i\xi\cdot x}$ for any $\xi \in \mathbb{R}^n$. They're not in L^2, and they form a continuum rather than a countable family. But the analogue of our $L^2(\Omega)$ basis expansions is the Fourier transform

$$f(x) = (2\pi)^{-n/2} \int e^{i\xi\cdot x} \hat{f}(\xi)\, d\xi$$

$$\hat{f}(\xi) = (2\pi)^{-n/2} \int e^{-i\xi\cdot x} f(x)\, dx$$

Since $\Delta(e^{i x\cdot\xi}) = -|\xi|^2 e^{i x\cdot\xi}$, soln of initial value pbm should be

$$u(x,t) = (2\pi)^{-n/2} \int e^{-|\xi|^2 t} \hat{u}_0(\xi) e^{i x\cdot\xi} \, d\xi.$$

$$= (2\pi)^{-n/2} \int e^{-|\xi|^2 t} u_0(y) e^{i x\cdot\xi} e^{-i(y\cdot\xi)} \, dy \, d\xi$$

$$= \int \Phi(x-y, t) \, u_0(y) \, dy$$

with

$$\Phi(z,t) = (2\pi)^{-n} \int e^{-|\xi|^2 t + i z\cdot\xi} \, d\xi.$$

Doing the integral: if $\eta = \xi\sqrt{t} - \frac{iz}{2\sqrt{t}}$ then

$$\Phi(z,t) = (2\pi)^{-n} \int e^{-|\eta|^2} e^{-|z|^2/4t} t^{-n/2} \, d\eta$$

$$= (2\pi)^{-n} e^{-|z|^2/4t} t^{-n/2} \underbrace{\int e^{-|\eta|^2} \, d\eta}_{\pi^{n/2}}.$$

$$= (4\pi t)^{-n/2} e^{-|z|^2/4t}$$

as expected.

Some features of this explicit soln formula:

- We easily see that soln is instantly C^∞, no matter how bad u_0 might be, by differentiating under the integral

- We see that ill-posedness of solving heat eqn backward in time is linked to difficulty of "deblurring" badly focused photos (blur from bad focus is similar to convolution with a Gaussian)

Can our "soln formula" in a bounded domain be put in a similarly explicit form, when the bc is either $u=0$ at $\partial\Omega$ or $\partial u/\partial n = 0$ at $\partial\Omega$? Yes indeed! Focusing on $u=0$ at $\partial\Omega$ to be specific, we have that

$$u_t - \Delta u = 0 \quad \text{in } \Omega \quad \text{(bounded)}$$
$$u = 0 \quad \text{at } \partial\Omega$$
$$u = u_0 \quad \text{at } t=0$$

$$\Rightarrow \quad u(x,t) = \int_\Omega G(x,y;t)\, u_0(y)\, dy$$

with

$$G(x,y;t) = \sum_{n=1}^{\infty} e^{-\lambda_n t} \varphi_n(x)\, \varphi_n(y)$$

This is just a rewrite of our prev soln formula (recall: φ_n are an orthonormal basis of eigenfns

$-\Delta\varphi_n = \lambda_n \varphi_n$ with $\varphi_n = 0$ at $\partial\Omega$). Note that G is symmetric in x + y, but no longer a function of x−y.

G(x, y; t) is called the "Green's function" of the heat eqn with bc $u=0$ at $\partial\Omega$. (There's an entirely analogous repn with bc $\partial u/\partial n = 0$ at $\partial\Omega$, obtained using eigenfns of Δ with that bc).

What about heat eqn in a half-space? I'll focus on the 1D case (a half line) to keep the notation simple, but the same ideas apply to $\mathbb{R}^n \cap \{x_n > 0\}$.

For Dir bc ($u=0$ at bdry) or Neumann bc ($\partial u/\partial n = 0$ at bdry) soln is easy, by reflection. Key pt is that

- if $u(x) = u(-x)$ then $u_x(0) = 0$ (assuming u is diffble)

- if $u(x) = -u(-x)$ then $u(0) = 0$ (assuming u is conts)

- by using either the odd or even

extension of the initial data we get a soln of heat eqn in all $\mathbb{R}^n$ that's correspondingly odd or even in x for all $t>0$.

Thus: $\quad u_t - u_{xx} = 0$ for $x > 0$, $\quad u = 0$ at $x = 0$

$$\Rightarrow \quad u(x,t) = \int_0^\infty G(x,y;t)\, u_0(y)\, dy$$

(*) $$G(x,y;t) = \Phi(x-y,t) - \Phi(x+y,t)$$

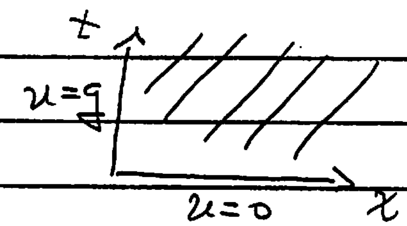
Similarly, $\quad u_t - u_{xx} = 0$ for $x > 0$, $\quad u_x = 0$ at $x = 0$.

$$\Rightarrow \quad u(x,t) = \int_0^\infty N(x,y;t)\, u_0(y)\, dy$$

$$N(x,y;t) = \Phi(x-y,t) + \Phi(x+y,t)$$

What about nonzero bdry data? I'll focus on 1D halfline. Case of halfspace in $\mathbb{R}^n$ or bounded $\Omega \subset \mathbb{R}^n$ is essentially the same. I'll do Dir bc, but Neumann bc can be done similarly (this will be HW). So consider:

$$u_t - u_{xx} = 0 \text{ for } x > 0$$
$$u = g(t) \text{ at } x = 0$$
$$u = 0 \text{ at } t = 0$$



<u>Claim</u>: soln is $u(x,t) = \int_0^t \frac{\partial G}{\partial y}(x,0;t-s)\, g(s)\, ds$

where G is the Green's function of the half-space problem with a homogeneous (0) Dir bc (formula (*) in pg 4.8). After some arithmetic this amounts to

$$u(x,t) = \int_0^t \frac{x}{\sqrt{4\pi}\,(t-s)^{3/2}} e^{-x^2/4(t-s)} g(s)\, ds,$$

<u>Proof</u>: fix (x_0,t_0) and consider

$$v(y,\tau) = G(x_0, y, t_0-\tau)$$

It solves heat eqn backward in time, with a δ-fn singularity at $y = x_0$ as $\tau \uparrow t_0$. So for $0<s<t_0$,

$$\frac{d}{ds}\int_0^\infty v(y,s)\, u(y,s)\, dy = \int_0^\infty v_s u + v u_s \, dy$$

$$= \int_0^\infty -(\Delta v)u + v(\Delta u)\, dy$$

$$= -v_y u \Big|_0^\infty + v u_y \Big|_0^\infty$$

The limits as $y \to \infty$ vanish (since $v \to 0$ and $v_y \to 0$ there) so

$$RHS = +v_y u\Big|_{y=0} - \cancel{v u_y}^{\,0}\Big|_{y=0} = g(s)\, G_y(x_0, 0; t_0 - s).$$

But:

$$\lim_{s \uparrow t_0} \int_0^\infty v(y,s)\, u(y,s)\, dy = u(x_0, t_0)$$

$$\lim_{s \downarrow 0} \int_0^\infty v(y,s)\, u(y,s)\, dy = 0$$ since by hypoth $u = 0$ at $t = 0$.

Thus by fund thm of calc

$$u(x_0,t_0) = \int_0^{t_0} \left(\frac{d}{ds} \int_0^\infty v(y,s)\, u(y,s)\, dy \right) ds$$

$$= \int_0^{t_0} G_y(x_0, 0; t-s)\, g(s)\, ds$$

as expected.