PDE - Lecture 3, 9/16/2014

[start with topics left over from Lecture 2 notes: representation formula for $u_t - \Delta u = f$; nonzero but time-independent bc; finite-difference approxn (discrete space, cont's time; or discrete space, discrete time)].

Another perspective: The evolution

$$
\begin{aligned}
u_t &= \Delta u && \text{in } \Omega \times (0, \infty) \\
u &= 0 && \text{at } \partial\Omega \\
u &= u_0(x) && \text{at } t = 0
\end{aligned}
$$

represents "steepest descent for $\frac{1}{2}\int_\Omega |\nabla u|^2$, among functions satisfying the Dir bc $u=0$ at $\partial\Omega$"

Before explaining this we need to warm up a bit

Warmup #1. Recall from ODE: given a function $F: \mathbb{R}^n \to \mathbb{R}$, the assoc "steepest descent" ODE is

$$\dot{z} = -\nabla F(z(t)) .$$

It has the property that $F(z(t))$ decreases (as t increases); in fact

$$\frac{d}{dt} F(z(t)) = \sum \frac{\partial F}{\partial z_i}(z(t))\, \dot{z}_i(t)$$

$$= \langle \nabla F(z(t)), \dot{z}(t) \rangle \quad \text{by defn of inner product}$$

$$= -|\dot{z}|^2$$

which is strictly neg unless initial cond is a critical pt.

Also note: in $\mathbb{R}^n$ we have $\nabla F = \left(\frac{\partial F}{\partial z_1}, \cdots, \frac{\partial F}{\partial z_n}\right)$ but a coord-free defn can also be given:

for any curve $z(\tau)$ (not solving any ODE)

$$\frac{d}{d\tau} F(z(\tau)) = \langle \nabla F(z(\tau)), \dot{z}(\tau) \rangle .$$

(With this defn we see that the meaning of ∇F depends on the choice of inner product; we get $\nabla F = (\partial F/\partial x_1, \cdots, \partial F/\partial x_n)$ only when we use the standard inner product on $\mathbb{R}^n$.)

Warmup #2: Still considering the ODE $\dot{z} = -\nabla F(z)$, there are two "obvious" ways to discretize it in time

"explicit Euler" $$\frac{z(t_{n+1}) - z(t_n)}{\Delta t} = -\nabla F(z(t_n))$$

or

"implicit Euler" $\quad \dfrac{z(t_{n+1}) - z(t_n)}{\Delta t} = -\nabla F(z(t_{n+1}))$

Explicit Euler is easy to implement: you read off soln at t_{n+1} from info at time t_n. It is what we used as our "discrete-time + discrete-space" scheme in Lecture 2 (pg 2.12). Alas, it can be unstable if the time step is too large (in Lecture 2, pg 2.13, the stability restrn was $\Delta t \le \frac{1}{2}(\Delta x)^2$).

Implicit Euler is harder to implement, because for general F we must solve a nonlinear eqn to find $z(t_{n+1})$. But actually it's not so bad: the time step problem is the EL eqn for the variational problem

$$y = z(t_{n+1}) \text{ solves } \min_y F(y) + \frac{|y - z(t_n)|^2}{2\Delta t}$$

(Note: if F is convex and $F \to \infty$ as $|y| \to \infty$ then this var'l pbm has a unique crit pt.)

Connection to heat eqn, 1st pass: Consider our discrete-space, cont's time version of 1D heat eqn with Dir bc

$$\dot{u}_j(t) = \frac{u_{j+1} + u_{j-1} - 2u_j}{(\Delta x)^2}$$

$$u_0(t) = 0 \text{ and } u_N(t) = 0 \text{ for all } t$$

$$u_j(0) = u_0(j\Delta x) \qquad j = 1, 2, \cdots, N-1.$$

This ODE is steepest descent for

$$F = \frac{1}{2}\sum_{j=1}^{N} \frac{|u_j - u_{j-1}|^2}{(\Delta x)^2}$$

viewed as function of $u_1, \cdots, u_{N-1}$, with the convention $u_0 = u_N = 0$.

Notice that $F \cdot \Delta x$ is a finite-difference approxn of

$$\frac{1}{2}\int_0^1 |\nabla u|^2 \, dx$$

<u>Connection to heat eqn, 2nd pass</u>: we now justify our assertion that $u_t - \Delta u = 0$ with $u|_{\partial\Omega} = 0$ is "steepest descent" for the "Dirichlet integral"

$$E[u] = \frac{1}{2}\int_\Omega |\nabla u|^2 \, dx$$

among functions st $u|_{\partial\Omega} = 0$, using the L^2 inner product.

In fact: if $v(\tau,x)$ is any (smooth enough) function at $v=0$ at $\partial\Omega$,

$$\frac{d}{d\tau} E[v(\tau)] = \int_\Omega (\nabla v, \nabla v_\tau)\, dx$$

$$= -\int_\Omega (\Delta v)\, v_\tau\, dx$$

(there's no bdry term since $v=0$ at $\partial\Omega \Rightarrow v_\tau = 0$ at $\partial\Omega$)

$$= \langle -\Delta v, v_\tau \rangle_{L^2(\Omega)}.$$

So $-\Delta v = \nabla E[v]$. (More standard terminology + notation in this function space setting: $-\Delta v$ is the "first variation of $E = \frac{1}{2}\int_\Omega |\nabla v|^2$ with bc $v|_{\partial\Omega} = 0$", typically written $-\Delta v = \frac{\delta E}{\delta v}$.)

From this perspective, we see that the "implicit" time discretization for heat eqn is

$$\frac{u(x,t_{n+1}) - u(x,t_n)}{\Delta t} = \Delta u(x,t_{n+1}) \quad \text{in } \Omega,$$

and that $u(x,t_{n+1})$ achieves the min of the variational problem

$$\min_{v=0 \text{ at } \partial\Omega} \frac{1}{2}\int_\Omega |\nabla v|^2 dx + \int_\Omega \frac{|v - u(x,t_n)|^2}{2\Delta t} dx$$

(We'll discuss such var'l pbms later, when we get to Laplace's eqn + related problems.)

The "implicit" time step for the discrete-time/discrete space pbm in 1D (with bc $u=0$ at endpts) is obviously

$$\frac{u_j(t_{n+1})-u_j(t_n)}{\Delta t} = \frac{u_{j+1}(t_{n+1})+u_{j-1}(t_{n+1})-2u_j(t_n)}{(\Delta x)^2}$$

and $\{u_j(t_{n+1})\}=\{v_j\}$ achieves

$$\min_{v_0=v_N=0} \frac{1}{2}\sum_{j=1}^{N} \frac{(v_j-v_{j-1})^2}{(\Delta x)^2} + \frac{1}{2}\sum_{j=1}^{N-1} \frac{|v_j - u_j(t_n)|^2}{\Delta t}.$$

A key advantage of this scheme over the explicit one: it is stable for any choice of Δt. (In general: the implicit Euler scheme is always stable.) See eg Strauss's chapter on numerical methods, for discn of this (in the context of the 1D heat eqn).

The "steepest descent" viewpoint works as well in the nonlinear setting as in the linear one (but: not every nonlin pde has a steepest-descent interpretation).

Some examples:

$$u_t = \Delta u + u^5 \quad \text{is steepest descent for} \quad \int_\Omega \tfrac{1}{2}|\nabla u|^2 - \tfrac{1}{6}u^6$$

and

$$u_t = \operatorname{div}\left(|\nabla u|^2 \nabla u\right) \quad \text{is steepest descent for} \quad \int_\Omega \tfrac{1}{4}|\nabla u|^4$$

where in all cases (for the pde and the var'l prbms) we use eg the bc $u=0$ at $\partial\Omega$. (The case of a Neumann bc $\partial u/\partial n = 0$ at $\partial\Omega$ will be discussed in the HW).

The steepest descent viewpt can even be used when the functional is not differentiable, eg

$$u_t = \operatorname{div}\left(\frac{\nabla u}{|\nabla u|}\right) \quad \text{is steepest descent for} \quad \int_\Omega |\nabla u|$$

but in the non-differentiable case the pde is purely formal, & the evolution is <u>defined</u> by implicit time stepping in the limit $\Delta t \to 0$. (See eg Evans §9.6 on "gradient flows," but note that the material there lies far beyond our syllabus.)