PDE - Lecture 2, 9/9/2014

[start with discn of max prin, left over from Lecture 1 notes]

Next focus: several ways of representing/approximating/thinking about solns of $u_t - \Delta u = 0$ (and related eqns) in bounded domains (with suitable bdry conds). We'll discuss

a) separation of variables

b) finite differences (cont's in time, discrete in space)

c) a steepest-descent interpretation (leading naturally to implicit-in-time discretization)

Recurrent themes:

1) linear heat eqn is, essentially, an "infinite-dimensional ODE" (indeed, a rather special one, similar to $\dot{x} = Ax$ where $x(t) \in \mathbb{R}^n$ and A is an $n \times n$ pos def symmetric matrix)

2) different viewpoints have different strengths (eg with regard to what's easy to see, and what types of generalization are possible).

<u>Separation of variables</u>. (Suggested reading: Guenther + Lee 3.5.1 + 5.3. It does only 1D problems, where eigenfunctions of Δ are explicit using $\sin nx$ or $\cos nx$, but nD case is no different if you accept completeness of eigenfunctions of Δ as a basis of $L^2(\Omega)$.)

Focus for now on

$$u_t - \Delta u = 0 \quad \text{in } \Omega \subset \mathbb{R}^n$$
$$u = 0 \quad \text{at } \partial\Omega$$
$$u = u_0(x) \quad \text{at } t = 0,$$

where Ω is a <u>bounded</u> domain. <u>Key fact</u>: the (normalized) eigenfunctions of Δ with a Dirichlet bc

$$-\Delta \varphi_n = \lambda_n \varphi_n \quad \text{in } \Omega$$
$$\varphi_n = 0 \quad \text{at } \partial\Omega$$

form a complete orthonormal set (ie they span $L^2(\Omega)$). Use of this basis <u>diagonalizes</u>

the operator Δ (with the Dirichlet bc).

Expanding initial data in this basis

$$u_0 = \sum{}' a_n \varphi_n(x)$$

with $a_n = \langle u_0, \varphi_n \rangle = \int u_0(x)\varphi_n(x)\,dx$ (I assume here that $\|\varphi_n\|_{L^2}^2 = \int \varphi_n^2(x)\,dx = 1$), we see immediately that

$$(*) \qquad u(x,t) = \sum{}' a_n e^{-\lambda_n t} \varphi_n(x)$$

solves the heat eqn.

The convergence of the sum, and the sense in which $\sum{}' a_n e^{-\lambda_n t}\varphi_n(x) \xrightarrow[t\to 0]{} u_0(x)$, have

some subtlety. For example: u_0 can be any L^2 function (it need not vanish at $\partial\Omega$). The special case $u_0(x) \equiv 1$ is instructive. See Guenther + Lee for good discn of these topics.

Notes:

(1) in 1D, if $\Omega = [0, L]$, $\varphi_n(x) = \sqrt{\frac{2}{L}}\sin\left(\frac{n\pi}{L}x\right)$, $\lambda_n = n^2\pi^2/L^2$, $n = 1, 2, \ldots$

Eigenvalues are simple in this case.

In 2D, if $\Omega = [0,L]\times[0,L]$, eigenfns are $\frac{2}{L}\sin\left(\frac{k\pi}{L}x\right)\sin\left(\frac{l\pi}{L}y\right)$ with assoc eigenvalue $(k^2+l^2)\pi^2/L^2$. Eigenvalues are not always simple in this case. (We usually order them s.t. $\lambda_1 \le \lambda_2 \le \lambda_3 \le \cdots$)

In general we cannot expect to have a formula for φ_n, but (*) still serves as a representation ("formula for") the solution of the pde.

(2) We see from (*) that u decays exp to 0 (since $\lambda_1 > 0$) and that

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |u|^2 \le -\lambda_1 \int_\Omega |u|^2$$

(recovering the result we obtained by an "energy argument" combined with Poincaré's ineq. with its sharp constant).

(3) We also see from (*) that solving heat eqn backward in time is ill-posed.
Put more precisely: soln of $v_t + \Delta v = 0$ for $t > 0$ is exquisitely sensitive to small changes in its initial data, since $v = \sum a_n e^{+\lambda_n t}\varphi_n(x)$

and higher modes (large n, so large λ_n) grow faster. So it is not true that $\|v_0 - \tilde{v}_0\|$ small (at time 0) $\Rightarrow$ $v - \tilde{v}$ small later on. (Such an est. is true, by max prin, when we solve the "correct direction" in time.)

Note: Though heat eqn is ill-posed when solved the "wrong dirn in time", for a given "initial data" there can be at most one soln. See Evans §2.3.4

(4) Situation is very much like $\dot{x} = Ax$ in $\mathbb{R}^N$, where A is a symmetric, pos def $N \times N$ matrix (which we can diagonalize using a basis of eigenvectors).

In fact, $u \to \Delta u$ is a self-adjoint linear map on the class of (smooth enough) functions u st $u|_{\partial\Omega} = 0$, with the L^2 inner product. To see this, just observe

$$\langle u, \Delta v \rangle = \int_\Omega u \Delta v = -\int_\Omega (\nabla u, \nabla v)$$

$$= \int_\Omega \Delta u \cdot v = \langle \Delta u, v \rangle$$

provided $u|_{\partial\Omega} = 0$ and $v|_{\partial\Omega} = 0$ (so the integn by parts has no bdry term).

Same calcn shows that $-\Delta$ is positive:

$$\langle u, -\Delta u \rangle = \int_\Omega |\nabla u|^2 > 0 \quad \text{unless } u \equiv 0$$

(using the bc $u|_{\partial\Omega} = 0$ again). Thus for eigenvalues of Δ with Dirichlet bc ($u|_{\partial\Omega}=0$) the eigenvalues λ_n are all positive.

(5) If eqn has nonzero RHS (a "source term"), no pblm.

coordinate-based version: to solve

$$\begin{aligned} u_t - \Delta u &= f(x,t) && \text{in } \Omega \\ u &= u_0 && \text{at } t=0 \\ u &= 0 && \text{at } \partial\Omega \end{aligned}$$

just expand $f = \sum' c_n(t)\,\varphi_n(x)$. Then

$$u = \sum{}' a_n(t)\,\varphi_n(x)$$

where $a_n(t)$ solves $\quad \dot{a}_n + \lambda_n a_n = c_n(t)$

$$a_n(0) = \langle u_0, \varphi_n \rangle$$

Less coordinate-bound version: recall from ODE that the sol'n of

$$\dot{z} = Az + \xi(t), \quad z(0) = z_0$$

in $\mathbb{R}^N$ (with A an $N\times N$ matrix) is

$$z(t) = e^{At} z_0 + \int_0^t e^{(t-s)A} \xi(s)\, ds.$$

(Proof: mult eqn by e^{-At} to get $(e^{-At}z)_t = e^{-At}\xi$, then integrate.) Analogous formula for

$$\begin{aligned} u_t - \Delta u &= f & t &> 0 \\ u &= 0 & &\text{at } \partial\Omega \\ u &= u_0 & &\text{at } t=0 \end{aligned}$$

is

$$(\ast\ast) \qquad u(\cdot,t) = e^{t\Delta} u_0(\cdot) + \int_0^t e^{(t-s)\Delta} f(\cdot,s)\, ds$$

where

$e^{t\Delta} g(\cdot)$ means "the solution of the lin heat eqn $v_\tau - \Delta v = 0$ with initial data $v = g$ at $\tau = 0$, evaluated at time t"

Equivalence of coord-bound + non-coord-bound versions: just represent $e^{t\Delta}$ using eigenfns of Δ.

Rmk: The repn (**) is especially useful for considering nonlinear eqns such as $u_t - \Delta u = f(u)$

(6) What about nonzero (Dirichlet-type) bc? We'll get an expression later that works for general, time-dependent bdry data. But in the time-independent case the answer is simple: use the linear character of the problem. Example: let's solve

$$u_t - \Delta u = 0 \qquad t > 0$$
$$u = u_0 \qquad \text{at } t = 0$$
$$u = 1 \qquad \text{at } \partial\Omega$$

Let $\tilde{u} = u - 1$. It solves

$$\tilde{u}_t - \Delta \tilde{u} = 0$$
$$\tilde{u} = u_0 - 1 \quad \text{at } t = 0$$
$$\tilde{u} = 0 \qquad \text{at } \partial\Omega$$

Since $u = \tilde{u} + 1$ we have

$$u = 1 + \sum_{n=1}^{\infty} \langle u_0 - 1, \varphi_n \rangle e^{-\lambda_n t} \varphi_n$$

L^2 inner product!

(Generalization of this: to solve $u_t - \Delta u = 0$ with bc $u = g$ at $\partial\Omega$, where g is indep of t, consider $\tilde{u} = u - U$ where $\Delta U = 0$ in Ω and $U = g$ at $\partial\Omega$.)

(7) Almost everything done above works equally well for Neumann bc ($\partial u/\partial n = 0$ at $\partial\Omega$)!

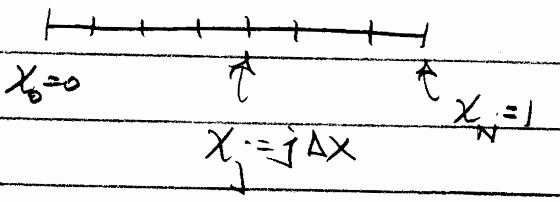
But: in this case you must use the eigenvalues + eigenfunctions of the Laplacian with a Neumann bc

$$-\Delta \psi_n = \lambda_n \psi_n \quad \text{in } \Omega$$
$$\partial\psi_n/\partial n = 0 \text{ at } \partial\Omega.$$

One difference: 1^{st} eigenvalue is zero (with ψ_1 = assoc eigenfunction = constant).

So: soln of $u_t - \Delta u = 0$, $\frac{\partial u}{\partial n} = 0$ at $\partial\Omega$ doesn't decay to zero, but rather to a constant (namely: the average value of $u_0(x)$).

Now a different perspective: finite difference approxn. For simplicity let's stick to 1D, say $\Omega = [0,1]$. (Extnsn to a square in 2D is obvious; extension to domains with curved boundaries is much more subtle.)



$\Delta x = \frac{1}{N}$

Obvious discretization (in space only) of $u_t = u_{xx}$ is ODE system for $u_j(t) = u(j\Delta x, t)$

$$\dot{u}_j = \frac{u_{j-1} + u_{j+1} - 2u_j}{(\Delta x)^2}$$

[Recall from Calc I: if $f(x)$ is smooth enough

$$f(x+\Delta x) = f(x) + \Delta x f'(x) + \tfrac{1}{2}(\Delta x)^2 f''(x) + \tfrac{1}{6}(\Delta x)^3 f'''(x) + O(|\Delta x|^4)$$

$$f(x-\Delta x) = f(x) - \Delta x f'(x) + \tfrac{1}{2}(\Delta x)^2 f''(x) - \tfrac{1}{6}(\Delta x)^3 f'''(x) + O(|\Delta x|^4)$$

so $\left| f''(x) - \dfrac{f(x+\Delta x) + f(x-\Delta x) - 2f(x)}{(\Delta x)^2} \right| \le C|\Delta x|^2$].

Bdry conds?

- if bc fixes $u_0(t) = u(0,t)$ and $u_N(t) = u(1,t)$

Then no problem: just solve for $u_1(t), \dots, u_{N-1}(t)$.

- if we have a homogeneous Neumann condn ($u_x = 0$ at $x = 0, 1$) then look for a soln that's <u>even</u> about the endpts, ie view $u_{-1}(t) = u_1(t)$ and $u_{N+1}(t) = u_{N-1}(t)$. So we can solve for $u_0(t)$ + $u_N(t)$ (as well as for u_j, $1 \le j \le N-1$) Eqns for u_0 + u_N are

$$\dot{u}_N = \frac{2u_{N-1} - 2u_N}{(\Delta x)^2} \qquad \dot{u}_0 = \frac{2u_1 - 2u_0}{(\Delta x)^2}$$

How accurate is this? Let's focus on case of Dir bc 0, + let's assume the exact soln is C^4 (uniformly in space + time). Let

$$z_j = u_j(t) - u^{exact}(j\Delta x, t)$$

where u^{exact} solves the pde. Then

$z_j = 0$ initially (by choice of numerical initial data)

$Z_0(t) = Z_N(t) = 0$ (by choice of bc)

and $$\dot{z}_j - \frac{z_{j+1} + z_{j-1} - 2z_j}{(\Delta x)^2} = O((\Delta x)^2).$$

There will be a HW pbm on why this implies the error estimate $|z_j(t)| \le C(\Delta x)^2 t$.

Note: while our separation of variables soln used linearity of pde in an essential way, the finite difference scheme just discussed extends straightforwardly to many nonlin eqns, eg $u_t - \Delta u = f(u)$.

What if we discretize both space + time? The simplest ("explicit Euler") scheme is

$$\frac{u_j(t_{n+1}) - u_j(t_n)}{\Delta t} = \frac{u_{j-1}(t_n) + u_{j+1}(t_n) - 2u_j(t_n)}{(\Delta x)^2}$$

with $t_n = n\Delta t$. Reorganization gives

$$u_j(t_{n+1}) = \alpha u_{j-1}(t_n) + \alpha u_{j+1}(t_n) + (1-2\alpha) u_j(t_n)$$

with $\alpha = \frac{\Delta t}{(\Delta x)^2}$. View this as

$u_j(t_{n+1})$ = weighted avg of u_{j-1}, u_j, u_{j+1} at time t_n

It is crucially important that $\boxed{\alpha \le 1/2}$, i.e. that $\Delta t \le \frac{1}{2}(\Delta x)^2$, so that all weights are nonneg. Then it's easy to see that

$$\max_j |u_j(t_{n+1})| \le \max_j |u_j(t_n)|$$

a sort of discrete max principle. If $\alpha > 1/2$ the weight $1-2\alpha$ is neg, the discrete max/min fails, and the scheme is unstable (eg small initial data can grow exponentially fast).

For a good (yet concise & elementary) discn of this stability condition see W. Strauss's book §8.2

["steepest descent" interpretation to be discussed in next set of notes]