PDE I - Lecture 13 - 12/9/2014

Final topic: Hamilton-Jacobi eqns.

Goals: understand

- Hopf-Lax solution formula for $u_t + H(\nabla u) = 0$;
- link btwn optimal control and 1st order pde

We'll also touch (briefly) on the notion of "viscosity solution" but we won't have much time for that.

Suggested reading: §3.3 and §10.3 of Evans (which cover all I'm discussing here, and *much* more).

Recall our discn of the role of (small) viscosity in selecting a special ("admissible") soln of Burgers' eqn:

- viscous perturbation is $u_t + uu_x = \varepsilon u_{xx}$, $u|_{t=0} = g$
- integrate once to get $w_t + \frac{1}{2}(w_x)^2 = \varepsilon w_{xx}$, $w|_{t=0} = h$
- change vars by $v = \exp(-w/2\varepsilon) \Rightarrow v$ solves the linear heat eqn $v_t - \varepsilon v_{xx} = 0$
- behavior of explicit solution formula as $\varepsilon \to 0$ can be studied using "Laplace asymptotics," leading to $w(x,t) = \min_y \left\{ \frac{|x-y|^2}{2t} + h(y) \right\}$

Today we'll gain a new perspective on this var'l principle for w (without Laplace asymptotics).

Our discn works most naturally for the final-value problem

(*) $$u_t + H(\nabla u) = 0 \text{ for } t < T, \quad u = g \text{ at } t = T.$$

where $x \in \mathbb{R}^n$ and $H: \mathbb{R}^n \to \mathbb{R}$. We'll need to assume H is convex or H is concave. (If you really want to solve an initial value problem, note that for $\tau = T - t$, $\tilde{u}(x,\tau) = u(x,t)$ solves $\tilde{u}_\tau - H(\nabla \tilde{u}) = 0$ with $\tilde{u} = g$ at $\tau = 0$.)

Optimal control interpn of (*), for H convex: consider

(**) $$u(x,t) = \max_{\alpha(s)} \left\{ \int_t^T h(\alpha(s))\, ds + g(y(T)) \right\}$$

$$\left[\begin{array}{l} \frac{dy}{ds} = \alpha(s) \\ y(t) = x \end{array} \right] \Leftarrow \text{"state eqn"}$$

where $h: \mathbb{R}^n \to \mathbb{R}$ is related to H by

$$H(p) = \max_{a \in \mathbb{R}^n} \{ a \cdot p + h(a) \}$$

or equivalently

$$-h(a) = \max_{p \in \mathbb{R}^n} \{ a \cdot p - H(p) \}$$

(example: $h(a) = -\frac{1}{2}|a|^2$, $H(p) = \frac{1}{2}|p|^2$; in general, $-h$ is the "Fenchel transform" of H and vice versa; note that h is concave.)

Claim 1: Since h is concave, an optimal path $y(s)$ should have const velocity, ie $\alpha(s)$ should be constant.

Pf: h concave $\Rightarrow$ by Jensen's ineq,

$$h[\text{avg velocity}] \geq \text{average of } h(\text{velocity})$$

Now, avg velocity $= \frac{1}{T-t} \int_t^T \frac{dy}{ds}\, ds = \frac{y(T) - y(t)}{T-t}$

$$\begin{aligned} \text{So } \int_t^T h(\alpha(s))\, ds &= (T-t) \cdot \text{avg of } h(\text{velocity}) \\ &\leq (T-t) \cdot h(\text{avg velocity}) \\ &= (T-t)\, h\left(\frac{y(T) - y(t)}{T-t} \right) \end{aligned}$$

= value assoc to const-velocity path with $\alpha = \frac{y(T) - y(t)}{T-t}$.

QED

<u>Conclusion</u>: The function u defined by $(**)$ has the simpler characterization

$$u(x,t) = \max_{z\in\mathbb{R}^n} \left\{ (T-t)\, h\!\left(\frac{z-x}{T-t}\right) + g(z) \right\}$$

This is the "Hopf-Lax solution formula" for the final-time problem $(*)$.

<u>Note</u>: the preceding was for convex H + final-value problem;

- as explained above, initial value pbm for $u_\tau - H(\nabla u) = 0$ with H convex is equivalent
- for the final-value pbm with concave H, must use min instead of max:

$$u(x,t) = \min_{\alpha(s)} \left\{ \int_t^T h(\alpha(s))\,ds + g(y(T)) \right\}$$

$$\begin{bmatrix} \frac{dy}{ds} = \alpha(s) \\ y(t) = x \end{bmatrix}$$

where
$$H(p) = \min_{a\in\mathbb{R}^n} \{ a\cdot p + h(a) \}$$

$$-h(a) = \min_{p\in\mathbb{R}^n} \{ a\cdot p - H(p) \}$$

(so h is convex; example: $H(p) = -\frac{1}{2}|p|^2$, $h(a) = \frac{1}{2}|a|^2$)

- for the initial value pbm $u_\tau - H(\nabla u) = 0$ with H concave, we use the last bullet with $\tau = T - t$.

End of digression, we return now to (**).

Claim 2: $u(x,t)$, defined by (**), solves (at least formally) the pde

$$u_t + H(\nabla u) = 0 \qquad t < T$$
$$u = g \qquad t = T$$

provided $H(p) = \max_{a \in \mathbb{R}^n} \{ a \cdot p + h(a) \}$

Explanation: Consider paths st $\frac{dy}{ds} = a$ (constant) briefly, for time Δt. Suppose that as $\Delta t \to 0$ they're suff't for optzn. Then

$$u(x,t) \simeq \max_a \{ h(a)\Delta t + u(x + a\Delta t, t + \Delta t) \}.$$

Now suppose is smooth enough to justify the following Taylor-expansion-based arg't:

$$u(x,t) \simeq \max_a \left\{ h(a)\Delta t + u(x,t) + a \cdot \nabla u(x,t)\Delta t + u_t(x,t)\Delta t \right\}$$

(up to errors of order $(\Delta t)^2$).

Cancel $u(x,t)$ from both sides + simplify $\Rightarrow$

$$0 \approx \max_a \{ h(a) + a \cdot \nabla u \} + u_t$$

i.e. $\quad 0 = u_t + H(\nabla u)$ with H defined as given above.

Example: $h(a) = -\frac{1}{2}|a|^2$ (concave) $\Rightarrow H(a) = \frac{1}{2}|a|^2$ (convex)
and we get "soln formula" for

(***) final-value pbm $u_t + \frac{1}{2}|\nabla u|^2 = 0$, $u(x,T) = g(x)$

in the form $\quad u(x,t) = \max_z \left\{ g(z) - \frac{|z-x|^2}{2(T-t)} \right\}$

Observations:

a) HJ eqn has infinitely many ae solns, only one of which agrees with our "soln formula"

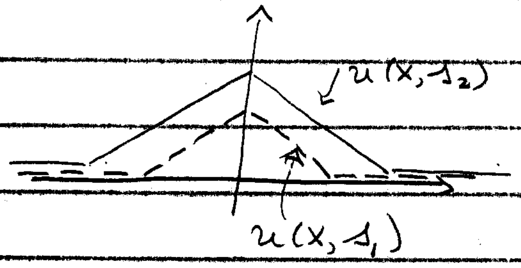
b) u given by soln formula can have discont's derivs (graph can have sharp valleys, when viewed as fn of x)

c) u given by soln formula cannot have sharp peaks

<u>About (a)</u>: Consider $g=0$, and 1D version of (***). Then soln formula gives $u=0$. But

$$u(x,t) = \begin{cases} \frac{1}{2}(T-t) - |x| & \text{if } |x| \le \frac{1}{2}(T-t) \\ 0 & \text{otherwise} \end{cases}$$

is also an ae soln

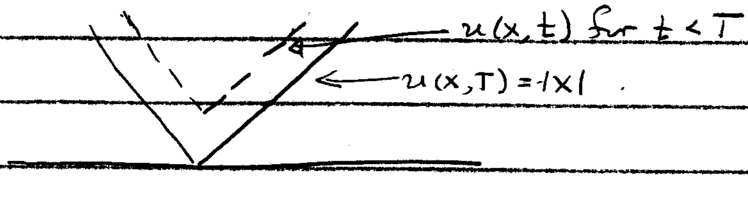


(Also: this artificial "bump" could be located anywhere - it need not be at $x=0$.)

<u>About (b)</u>: Consider the same example (1D version of (***)) but with $g(y)=|y|$. Then soln formula is easy to evaluate: best z is $x+(T-t)\,x/|x|$, yielding

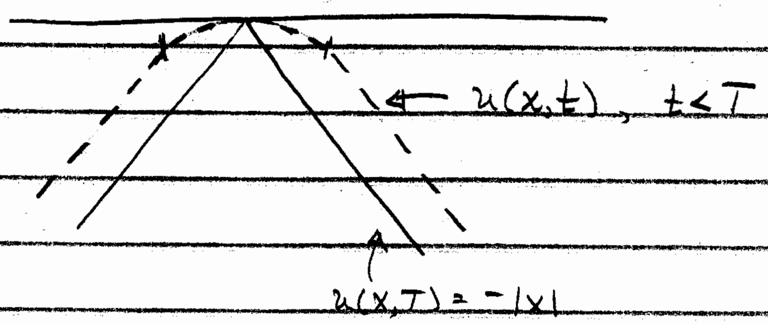
$$u(x,t) = \frac{T-t}{2} + |x|$$

(best z is nonunique at $x=0$, where u is singular)



Concerning (c): Still considering 1D version of (***), try $g(y) = -|y|$. Then best z is $x - (T-t)\, x/|x|$ if $|x| > T-t$ but $z = 0$ otherwise. Get

$$u(x,t) = \begin{cases} \frac{T-t}{2} - |x| & |x| \geq T-t \\ \dfrac{-|x|^2}{2(T-t)} & |x| \leq T-t \end{cases}$$



Evidently: HJ eqn $u_t + H(\nabla u) = 0$ has problem of nonuniqueness, like scalar cons laws. We need a selection criterion.

Fact: small viscosity and the optimal control interpretation make the same selection. Proof is in Evans §10.3 & lies beyond scope of this class. But let's show that as $\varepsilon \to 0$, soln of

$$u^\varepsilon_t + \tfrac{1}{2}(u^\varepsilon_x)^2 + \varepsilon u_{xx} = 0 \quad \text{for } t < T$$
$$u^\varepsilon = g \quad \text{at } t = T$$

cannot have a sharp peak (cf observation (c) above). If it did then u^ε would have a rounded peak, & at the top

$$u^\varepsilon_x = 0 \qquad (u^\varepsilon \text{ is smooth for } \varepsilon > 0!)$$
$$u^\varepsilon_{xx} \le 0$$

$$\Rightarrow u^\varepsilon_t \ge 0 \quad \text{at the peak.}$$

But if $u = \lim_{\varepsilon \to 0} u^\varepsilon$ has a "sharp" peak (discont'y in u_x) then $u_t = -\frac{1}{2} u_x^2 < 0$ at the peak. Contradiction.

Theory of "viscosity solns" gives a charzn of soln of
$$u^\varepsilon_t + H(\nabla u^\varepsilon) + \varepsilon \Delta u^\varepsilon = 0 \qquad t < T$$
$$u^\varepsilon = g \quad \text{at } t = T$$

in limit $\varepsilon \to 0$ (without actually introducing ε, and without a soln formula — for example it works the same even if H is neither convex nor concave). See Evans.

Preceding descr is not easy to visualize. Here is one that's easier: fix a "target set" $G \subset \mathbb{R}^n$. For $x \notin G$, consider "min arrival time"

$$u(x,t) = \min_{\left[\begin{array}{l} y(0)=x \\ \frac{dy}{ds} = f(y(s),\alpha(s)) \end{array}\right]\text{"state eqn"}} \{1^{st} \text{ time } y(s) \text{ arrives at } G\}$$

$$\alpha(s) \in A \quad \leftarrow \text{"admissible controls"}$$

Formal HJ eqn (now time indep) is

$$u(x) \simeq \min_{a \in A} \{\Delta t + u(x + f(x,a)\Delta t)\}$$

$$\simeq \min_{a \in A} \{\Delta t + u(x) + f(x,a)\cdot \nabla u(x) \cdot \Delta t\}$$

$$\Rightarrow \quad 0 = \min_{a \in A} \{f(x,a)\cdot \nabla u(x)\} + 1.$$

Special case: use state eqn $dy/ds = \alpha(s)$
and impose constraint $|\alpha(s)| \leq \varphi(y(s))$.
Then preceding calcn gives

$$\min_{|a| \leq \varphi(x)} \nabla u \cdot a + 1 = 0$$

i.e.

$$-\varphi(x)|\nabla u| + 1 = 0$$

Thus: $\quad |\nabla u| = 1/\varphi(x) \quad$ for $x \notin G$.

$$u = 0 \quad \text{at } G.$$

Special case $\varphi \equiv 1$ is familiar "eikonal eqn"

$$|\nabla u| = 1 \quad \text{for } x \notin G.$$
$$u = 0 \quad \text{at } G$$

In this case the best paths are straight lines (which are characteristics for the pde!)