Final topic: Hamilton–Jacobi eqns.

Goals: understand

- Hopf–Lax solution formula for $u_t + H(u_x) = 0$;
- Link between optimal control and 1st order pole.

We'll also touch (briefly) on the notion of "viscosity solution," but we won't have much time to that.

Suggested reading: 2.3.3 and 2.10.3 of Evans (which cover all I'm discussing here, and much more).

Recall our discussion of the role of (small) viscosity in selecting a special ("admissible") solution of Burgers' eqn:

- Viscous perturbation is $u_t + uu_x = E u_{xx}$, $u^1 = \phi$.
- Integrate once to get $w_t + \frac{1}{2} (w^2) = E w_{xx}$, $w^1 = \phi$.
- Change vars by $v = \exp \left(-\frac{w}{2E}\right)$ to $v$ solve the linear heat eq $v_t - E v_{xx} = 0$.
- Behavior of explicit solution formula as $t \to 0$ can be studied using "boundary asymptotics."

leading to $w(x,t) = \min \left\{ \frac{1}{2t} (x-y)^2 + E (x-y) \right\}$.
Today we'll gain a new perspective on the wave principle for $u$ (without Laplace asymptotes).

Our discussion works most naturally for the \textit{final-value problem}:

$$u_t + H(\nabla u) = 0 \text{ for } t < T, \quad u = g \text{ at } t = T,$$

where $x \in \mathbb{R}^n$ and $H : \mathbb{R}^n \to \mathbb{R}$. We'll need to assume $H$ is convex or $H$ is concave. (If you really want to solve an \textit{initial value problem}, note that for $T = T + t$, $\tilde{u}(x, T) = u(x, t)$ solves $\tilde{u}_t + H(\nabla \tilde{u}) = 0$ with $\tilde{u} = g$ at $t = 0$.)

Optimal control interpretation of (1): for $H$ convex, consider

$$u(x, t) = \max \left\{ \int_0^T \left[ p(x(s), a(s)) \, ds + g(y(T)) \right] \right\}_{a \in \mathbb{R}^n},$$

where $p : \mathbb{R}^n \to \mathbb{R}$ is related to $H$ by

$$H(p) = \max_{a \in \mathbb{R}^n} \left\{ a \cdot p + H(a) \right\}.$$
or equivalently

\[-p(a) = \max_{p \in \mathbb{R}^n} \left\{ a \cdot p - H(p) \right\} \]

(examples: \( p(a) = -\frac{1}{2} a^2 \), \( H(p) = \frac{1}{2} |p|^2 \). In general, -p is the "Fenchel transform" of H and vice versa; note that \( H \) is concave.)

Claim 1: Since \( H \) is concave, an optimal path \( y(t) \) should have constant velocity, i.e. \( \dot{y}(t) \) should be constant.

pf: \( H \) concave \( \Rightarrow \) by Jensen's ineq,

\[ \langle \text{avg velocity} \rangle \geq \text{average of } H(\text{velocity}) \]

Now, avg velocity \( \dot{y}(t) = \frac{1}{T-t} \int_{T-t}^{T} \frac{dy}{dt} \, dt = \frac{y(T) - y(t)}{T-t} \)

So \( \int_{T-t}^{T} \dot{y}(s) \, ds = (T-t) \cdot \text{avg of } H(\text{velocity}) \)

\[ \leq (T-t) \cdot H(\text{avg velocity}) \]

\[ = (T-t) \cdot H \left( \frac{y(T) - y(t)}{T-t} \right) \]

= value assoc to const. velocity path with \( \dot{y} = \frac{y(T) - y(t)}{T-t} \)

Q.E.D.
Conclusion: The function defined by (2.16) has the simpler characterization

$$u(x, t) = \max_{z \in \mathbb{R}^n} \left\{ (1 - t) h \left( \frac{2 - x}{1 + t} \right) + q(z) \right\}$$

This is the "Hopf-Lax solution formula" for the first-order problem (2.16).

Note: the procedure was for convex $H$ in the first-order problem:

- as explained above, initial value problem for
  $$u_t - H(u(x_t)) = 0$$
  with $H$ convex is equivalent

- for the final-value problem with concave $H$, must use $\min$ instead of $\max$:

$$u(x, t) = \min_{x(t)} \left\{ \frac{1}{t} \int_0^t h(x(s)) \, ds + g(y(t)) \right\}$$

\[ \begin{align*}
\frac{dx}{dt} &= x(t) \\
y(t) &= x
\end{align*} \]

where

$$H(p) = \min_{a \in \mathbb{R}} \left\{ a \cdot p + h(a) \right\}$$

$$-h(a) = \min_{p \in \mathbb{R}} \left\{ a \cdot p - H(p) \right\}$$

(assuming $h$ is convex; example: $H(p) = -\frac{1}{2} p^2$, $h(a) = \frac{1}{2} a^2$)
For the initial value problem \( u_t + H(Ju) = 0 \) with \( H \) concave, we use the last bullet with \( t = T - t \).

End of discussion, we return now to (18).

Claim 2: \( u(x,t) \), defined by (18), solves (at least formally) the pole

\[
\begin{align*}
    \frac{\partial u}{\partial t} + H(Ju) &= 0 \quad &t < T \\
    u &= g \quad &t = T
\end{align*}
\]

provided \( H(p) = \max_{a \in \mathbb{R}} \{ a \cdot p + b(a) \} \).

Explanation: Consider paths \( x + \frac{\partial y}{\partial t} = a \) (constant) briefly, for the \( a \). Suppose that as \( a \to 0 \) they're made to opt for. Thus

\[
u(x,t) = \max_a \{ b(a) a t + u(x + a(t, t + t)) \}.
\]

Now suppose is smooth enough to justify the following Taylor expansion-based arg t:

\[
u(x,t) = \max_a \left\{ b(a) a t + u(x,t) + a \cdot \nabla u(x,t) a t \right\} + u_t(x,t) a t.
\]
(up to errors of order $(\delta t)^2$).

Cancel $u(x,t)$ from both sides + simplifying $\Rightarrow$

$$0 = \max_a \left\{ \frac{\partial}{\partial a} \cdot a \cdot 7u^2 + 2u \right\}$$

$x$ $\Rightarrow$

$$0 = u_t + H(7u), \text{ with } H \text{ defined as given above.}$$

Example: $b(a) = -\frac{1}{2} 1a^2$ (concave) $\Rightarrow H(a) = \frac{1}{2} 1a^2$ (convex) and we get "sola formula" $f$

(yielding a fixed point) $u_t + \frac{1}{2} 17u^2 = 0 \Rightarrow u(x,t) = g(x)$

in the form $u(x,t) = \max_x \left\{ g(x) - \frac{1 - x^2}{x(t-t)} \right\}$

Observations:

a) HT eqn has infinitely many $u(x,t)$, only one of which agrees with our "sola formula."

b) $u$ given by sola formula can have discontinuous (graph can have sharp
drives when viewed as $u(x)$.)
(a) \( u \) given by \(-1 \leq u \leq 1\) formula cannot have steep peaks.

About (b): Consider \( f = 0 \), and 1D version of (**). Then the formula gives \( u = 0 \). But:

\[
    u(x,t) = \begin{cases} 
        \frac{1}{2} (t-x) - 1|x| & \text{if } 1|x| \leq \frac{1}{2} (T-t) \\
        0 & \text{otherwise}
    \end{cases}
\]

is also an ae soln.

(Also: this artificial "bump" could be located anywhere - it need not be at \( x=0 \).)

About (b): Consider the same example (1D version of (**)) but with \( f(y) = 1/4 \). Then the formula is easy to evaluate: best \( z \) is \( x + (T-t) \chi_{|x|} \), yielding:

\[
    u(x,t) = \frac{T-t}{2} + 1|x|
\]

(but \( z \) is nonunique at \( x=0 \), where \( u \) is singular.)
Concerning (c): Still considering 1D version of (2.44),
try \( g(y) = -|y| \). Then best \( z \) is \( x - (T-t) \frac{|x|}{|x| + 1} x \) for \( |x| > T-t \)
but \( z = 0 \) otherwise. Get:

\[
\begin{align*}
  u(x,t) &= \begin{cases} 
  \frac{T-t}{2} - |x| & |x| > T-t \\
  -\frac{|x|^2}{2(T-t)} & |x| \leq T-t 
  \end{cases}
\end{align*}
\]

\( u(x,t) = -|x| \) for \( t < T \).

Evidently: HT can \( v_4 + H(\vec{x}_4) = 0 \) for problem 1
consequences, like scalar conservation. We need a selection criterion.

Fact: Small viscosity, and the optimal control interpretation make the same selection. Proof
in Evans 3/10.3+ has beyond scope of this class. But let's show that as \( \varepsilon \to 0 \), solve:

\[
\begin{align*}
  u_\varepsilon^\varepsilon + \frac{1}{2} (u_\varepsilon^\varepsilon)_x^2 + \varepsilon u_\varepsilon = 0 & \quad \text{for } t < T \\
  u_\varepsilon^\varepsilon = g & \quad \text{at } t = T
\end{align*}
\]
cannot have a sharp peak (of observation (c) above). If it did then $u^\varepsilon$ would have a rounded peak, $\Phi$ at the top.

$$u^\varepsilon_x = 0 \quad (u^\varepsilon \text{ is smooth for } \varepsilon > 0!)
\quad u^\varepsilon_{xx} \leq 0$$

$$\Rightarrow u^\varepsilon_\Phi > 0 \text{ at the peak}.$$ But if $u = \lim_{\varepsilon \to 0} u^\varepsilon$ has a "sharp" peak (discontinuity in $u_x$) then $u_\Phi = -\frac{1}{\varepsilon} u^2 < 0$ at the peak. Contradiction.

Theory of "viscosity solutions" gives a chance of solving

$$u_t^\varepsilon + H(u^\varepsilon) + \varepsilon \Delta u^\varepsilon = 0 \quad 0 < t < T$$

$$u^\varepsilon = g \quad \text{at } t = T$$

in limit $\varepsilon \to 0$ (without actually introducing $\varepsilon$, and without a soln formula — for example it works the same even if $H$ is neither convex nor concave). See Evans.

Preceding lesson is not easy to visualize. Here is one that's easier: for a "target set" $G \subset \mathbb{R}^n$. For $x \notin G$, consider "min arrival time"
\[ u(x,t) = \min_{y(0) = x} \left\{ 1 + \text{the } y(0) \text{ arrives at } G \right\} \]

Formal HT eqn (now the u(x,y)) is

\[ u(x) = \min_{a \in A} \left\{ A + u(x + f(x,a) \cdot A) \right\} \]

\[ = \min_{a \in A} \left\{ A + u(x) + f(x,a) \cdot g u(x) \cdot A \right\} \]

\[ \Rightarrow 0 = \min_{a \in A} \left\{ f(x,a) \cdot g u(x) \right\} + 1. \]

Special case: case of the eqn \( \frac{dy}{dt} = x(t) \)

and impose constraint \( |x(t)| \leq g(y(t)) \).

Then preceding calculus gives

\[ \min_{a \in A} g u(x) + 1 = 0 \]

\[ |a| \leq g(x) \]

\[ \Rightarrow -g(x) |g u + 1 = 0 \]

Thus:

\[ |g u + 1 = \sqrt{g(x)} \quad \text{for } x \in G. \]

\[ u = 0 \quad \text{at } G. \]
Special case $\theta = 1$ is familiar "homology eqn."

$|q| = 1$ for $x \in G$.

$v = 0$ at $G$.

In this case the best paths are straight lines (which are characteristics for the pole!).