These notes:

1. Method of characteristics (for 1st-order PDEs)
2. Notion of a characteristic curve (for 2nd-order PDEs in 2-space dimensions)

Method of characteristics is a scheme for reducing the task of solving a broad class of 1st-order PDEs to solving a set of ODEs. (Our class $u_t + c(u)u_x = 0$ was a special case.) Guenther+Lee chapter 2 is a good source. (Evans does this too, of course.)

I'll work in $\mathbb{R}^2$ to simplify notation (the analogous theory in $\mathbb{R}^n$ requires no new ideas).

Linear case first:

$$a(x,y)u_x + b(x,y)u_y = c(x,y)u + d(x,y)$$

Eqn reduces to ODE along $x = x(s), y = y(s)$ if

$$\frac{dx}{ds} = a(x,y), \quad \frac{dy}{ds} = b(x,y)$$

Since then

$$\frac{d}{ds} u(x(s), y(s)) = c(x(s), y(s)) u(x(s), y(s)) + d(x(s), y(s))$$
Geometrically: \( \mathbf{E} = (x, y) \) is a vector field.

Eq. 1 says
\[ \mathbf{E} \cdot \nabla u = cu + d. \]

Appropriate "initial data": specify \( u \) on a curve in \( \mathbb{R}^2 \) that's transverse to the vector field.

Local existence is clear. **Global existence can fail**: e.g., \( x u_x + y u_y = 0 \) with \( u \) specified on \( x^2 + y^2 = 1 \). Then \( u = \text{const} \) along each ray through \( 0 \), but it (usually) won't be differentiable at \( 0 \).

Next: quasi-linear case, namely
\[ a(x,y,u) u_x + b(x,y,u) u_y = c(x,y,u). \]

Similar method works, but now eqn of the curve depends on \( u \). We must solve...
\[ \frac{dx}{dt} = a(x, y, z), \quad \frac{dy}{dt} = b(x, y, z), \quad \frac{dz}{dt} = c(x, y, z) \]

(an autonomous system of ODE's). Along such a curve \( z = u(x, y) \), where \( u \) solves the ODE.

Thus to get a local solution of the ODE in \( \mathbb{R}^2 \), we can specify values of \( u \) along a curve \( S \subseteq \mathbb{R}^2 \).

Then solve ODE written above (\( S = 0 \) corresponds to a pt of \( S \)). Method works if \( (a(x, y, u), b(x, y, u)) \) vector to \( S \) is not tangent to \( S \).

Effectively: graph of \( u \) is obtained as union of solutions of ODE's. Solution is of course only local (it may break down, as shown in linear case).

Does the \( u(x, y) \) obtained this way actually solve the ODE? \( \sqrt{\text{Yes}} \). Proof uses implicit function theorem (not trivial, but see one of the books for this.)

Finally, the nonlinear case:

\[ F(x, y, u, u_x, u_y) = 0 \]

Use notation \( F = F(x, y, u, p, q) \).
Suppose we have a sub $u$. Consider the vector field in $\mathbb{R}^2$:

$$
\vec{v} = \left( \frac{\partial F}{\partial p}, \frac{\partial F}{\partial q} \right)
$$

$$
p = u_x(x, y), \quad q = u_y(x, y)
$$

Its integral curves satisfy

$$
\frac{dx}{dt} = \frac{\partial F}{\partial p}, \quad \frac{dy}{dt} = \frac{\partial F}{\partial q}
$$

so along such a curve

$$
\frac{d}{ds} u(x(s), y(s)) = u_x \dot{x} + u_y \dot{y}
$$

$$
= u_x \frac{\partial F}{\partial p} + u_y \frac{\partial F}{\partial q}
$$

In general, the case $\frac{\partial F}{\partial p} + \frac{\partial F}{\partial q}$ were just functions of $x, y, u$ alone, but now they depend on $p + q$. So we need to track them using ODE's. Evidently (since $p = u_x, \quad q = u_y$) we have

$$
\frac{dp}{ds} = u_{xx} \dot{x} + u_{xy} \dot{y} = u_{xx} \frac{F_p}{\delta} + u_{xy} \frac{F_q}{\delta}
$$

$$
\frac{dq}{ds} = u_{xy} \dot{x} + u_{yy} \dot{y} = u_{xy} \frac{F_x}{\delta} + u_{yy} \frac{F_y}{\delta}
$$

From $F(x, y, u, u_x, u_y) = 0$ we get
\[ 0 = F_x + F_u u_x + F_p u_x + \frac{F}{\delta} u_{xy} \]

\[ 0 = F_y + F_u u_y + F_p u_{xy} + \frac{F}{\delta} u_{yy} \]

So

\[ \frac{dp}{dx} = -F_x - F_u u_x = -F_x - F_u p \]

\[ \frac{d\delta}{dx} = -F_y - F_u u_y = -F_y - F_u \delta \]

Thus we can recover solutions of PDE restricted to a suitable curve by solving the ODE system

\[ \begin{align*}
    \dot{x}(s) &= F_p \\
    \dot{y}(s) &= F_p \\
    \dot{u}(s) &= F_p + \frac{F}{\delta} F_e \\
    \dot{p}(s) &= -F_x - F_u p \\
    \dot{\delta}(s) &= -F_y - F_u \delta
\end{align*} \]

(where RHS is evaluated at \( (x(s), y(s), u(s), p(s), \delta(s)) \)).

To get a (local) solution \( u(x,y) \) near a curve \( S \subset \mathbb{R}^2 \) we proceed as before: suppose \( S \) is parameterized by \( x = x(t) \), \( y = y(t) \) and we're given \( u \) along \( S \), i.e. \( u(x(t), y(t)) = g(t) \) is given. Then clearly specify the initial data for \( x(s), y(s), u(s) \) but what about \( p(s) + \delta(s) \)? Well, we know
\[
\begin{align*}
\phi \equiv & \quad \mathbf{u}_t = \alpha \mathbf{u}_x + \beta \mathbf{u}_y + \gamma, \quad \psi \equiv \mathbf{p}_t = \beta \mathbf{u}_x + \gamma \mathbf{u}_y, \\
0 = & \quad F(x, \mu, \phi, \psi, \chi).
\end{align*}
\]

In quasilinear case this was a linear system for \(p + \phi\) (at \(s = 0\)) + linear algebra told us whether it had a unique soln or not. In nonlinear case we must simply assume that

(1) admits a unique soln for \(p \geq 6\) and furthermore

(\(\phi_t, \psi_t\)) is not parallel to (\(F_\phi, F_\psi\))

With this condn ODE is solvable for each \(t\) (for \(s \in \text{nbhd of 0}\)).

As in quasilinear case: the fact that the \(u(x,y)\) built up this way does solve the pde is an appln of Implicit Function Theorem (see the books).

Now a word abt classification of 2nd order linear pde. (Why? I want to introduce the other usage of the word "characteristic".) Consider

\[
a(x, y) u_{xx} + 2b(x, y) u_{xy} + c(x, y) u_{yy} + d(x, y) u_x + e(x, y) u_y + f(x, y) u = g(x, y).
\]
It is **elliptic** if \( ac - b^2 > 0 \)

**hyperbolic** if \( ac - b^2 < 0 \)

**parabolic** if \( ac - b^2 = 0 \)

(Notice that if \( a, b, c \) are functions of \( x, y \) then type can be different at different points.)

**Question**: when can we specify \( u + \frac{\partial u}{\partial t} \) along a curve \( \delta \) in \((x, y)\) plane \( \delta \) by fully determine (at least formally) the function \( u \) near \( \delta \)?

\[
\delta = (\delta_1(t), \delta_2(t))
\]

**Idea**: use eqn to solve for 2nd always along \( \delta \)
if possible.

Know \( \frac{\partial u}{\partial x} \) \( \frac{\partial u}{\partial y} \) (since \( u + \frac{\partial u}{\partial t} \) determine the entire gradient \( \nabla u \) along \( \delta \))

By chain rule

\[
\frac{\partial u}{\partial s} = u_{xx} \delta_x'(s) + 2 u_{xy} \delta_x' \delta_y'(s)
\]

\[
\frac{\partial u}{\partial t} = u_{xy} \delta_y'(s) + 2 u_{yy} \delta_y' \delta_y'(s)
\]
So we have 3 linear equations in 3 unknowns \( u_{xx}, u_{xy}, u_{yy} \) at each point \( x' \):

\[
\begin{align*}
&a u_{xx} + 2b u_{xy} + c u_{yy} = \text{known} \\
b' u_{xx} + c' u_{xy} = \text{known} \\
c' u_{xy} + d' u_{yy} = \text{known}
\end{align*}
\]

These determine \( u_{xx}, u_{xy}, u_{yy} \) uniquely if the determinant is not zero. That is, the determinant is

\[
\det \begin{bmatrix}
a & b & c \\
b' & c' & 0 \\
0 & a' & b'
\end{bmatrix} = a(x_{1}'^2 + c(x_{1}'){')^2 - 2bcx_{1}'x_{1}'
\]

Note: normal to \( x' \) is \((x_{1}', -x_{2}') = 0\)

\[
\det \neq 0 \iff a \eta_{1}^2 + 2b \eta_{2} \eta_{1} + c \eta_{2}^2 \neq 0
\]

where \( \eta = \frac{x_{1}}{x_{2}} \)

A curve where \( \det \neq 0 \) is called noncharacteristic. One where \( \det = 0 \) (pointwise) is called characteristic. Evidently:

- elliptic \( \iff \) every curve is noncharacteristic
- hyperbolic \( \iff \) two distinct families of characteristic curves
- parabolic \( \iff \) just one family of characteristic curves.
Notes:

(1) For wave eqn, the characteristic curves are of course the lines \( x - t = \text{const} \) and \( x + t = \text{constant} \).

(2) For heat eqn \( u_t - u_{xx} = 0 \) the char curves are \( t = \text{constant} \) (and indeed, we cannot specify both \( u + u_t \) independently at \( t = 0 \)).

(3) Extension of this discus to 2nd order pole in \( \mathbb{R}^n \) leads to question: do \( u + \frac{\partial u}{\partial t} \) along a codim 1 surface \( S \) permit us to formally solve for all 2nd orders of \( u \) along \( S \)? If \( \phi \) in \( \sum \frac{\partial^2 u}{\partial x_i \partial x_j} + l o t s = 0 \) then answer is yes \( (\sum \frac{\partial^2 u}{\partial x_i \partial x_j} \) is formally determined) if \( \sum a_i \partial_i u \neq 0 \) where \( \vec{n} = \vec{n}_i \partial_i \) is normal vector to \( S \). We say \( S \) is a non-characteristic surface.

Evidently: for 1st order eqns, method of characteristics leads to curves (in \( xy \) space for this eqns, but in \( x,y,u,p,q \) space for nonlinear case), But for 2nd order eqns, preceding discus leads to codim -1 surfaces being characteristic or non-characteristic.
Q: When S is non-characteristic, so D^2 is formally determined, what about higher derivs? And they're all formally determined.

Q: Does this mean there's actually a local soln u? Ans: no, not in general. But yes if data + S are analytic (this is the Cauchy-Kovalevskaya Theorem)