# PDE - Lecture 12, 12/2/2014

**These notes:**

(1) method of characteristics (for 1st order eqns)

and

(2) notion of a characteristic curve (for 2nd order eqns in 2 space dimensions)

Method of characteristics is a scheme for reducing task of solving a broad class of 1st order eqns to soln of ODE's. (Our discn of $u_t + c(u)u_x = 0$ was a special case.) Guenther + Lee chap 2 is a good source. (Evans does this too, of course!)

I'll work in $\mathbb{R}^2$ to simplify notation (The analogous discn in $\mathbb{R}^n$ requires no new ideas).

Linear case first:

$$a(x,y)\,u_x + b(x,y)\,u_y = c(x,y)\,u + d(x,y)$$

Eqn reduces to ODE along $x = x(s)$, $y = y(s)$ if

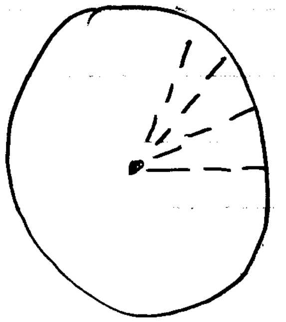
$$\frac{dx}{ds} = a(x,y), \qquad \frac{dy}{ds} = b(x,y)$$

since then
$$\frac{d}{ds}u(x(s),y(s)) = c(x(s),y(s))\,u(x(s),y(s)) + d(x(s),y(s))$$

Geometrically: $\xi = (a, b)$ is a vector field.
Eqn says
$$\xi \cdot \nabla u = cu + d$$

Appropriate "initial data": specify $u$ on a curve in $\mathbb{R}^2$ that's transverse to the vector field.

Local existence is clear. <u>Global existence can fail</u>: eg if $xu_x + yu_y = 0$ with $u$ = specified data on $x^2+y^2=1$ then $u$ = const along each ray through $0$, but it (usually) won't be differentiable at $0$



Next: <u>quasilinear case</u>, namely
$$a(x,y,u)\, u_x + b(x,y,u)\, u_y = c(x,y,u)$$

Similar method works, but now eqn of the curve depends on $u$. We must solve

$$\frac{dx}{ds} = a(x,y,z), \quad \frac{dy}{ds} = b(x,y,z), \quad \frac{dz}{ds} = c(x,y,z)$$

(an autonomous system of ODE's). Along such a curve $z = u(x,y)$ where $u$ solves the pde.

Thus: to get a local soln of pde in $\mathbb{R}^2$, we can specify values of $u$ along a curve $S \subset \mathbb{R}^2$. Then solve ODE written above ($s=0$ corresp to a pt of $S$). Method works if $(a(x,y,u), b(x,y,u))$ restr to $S$ is not tangent to $S$.

Effectively: graph of $u$ is obtained as union of solns of ODE's. Soln is of course only local (it may break down, as shown in linear case).

Does the $u(x,y)$ obtained this way actually solve the pde? Yes. Proof uses implicit function theorem (not trivial, but see one of the books for this.)

---

Finally the nonlinear case:

$$F(x,y,u,u_x,u_y) = 0$$

Use notation $F = F(x,y,u,p,q)$.

Suppose we have a soln $u$. Consider the vector field in $\mathbb{R}^2$:

$$\xi = \left(\frac{\partial F}{\partial p}, \frac{\partial F}{\partial q}\right)\Bigg|_{p = u_x(x,y),\ q = u_y(x,y)}$$

Its integral curves satisfy

$$\frac{dx}{ds} = \frac{\partial F}{\partial p}, \quad \frac{dy}{ds} = \frac{\partial F}{\partial q}$$

so along such a curve

$$\frac{d}{ds} u(x(s), y(s)) = u_x \dot{x} + u_y \dot{y}$$
$$= u_x \frac{\partial F}{\partial p} + u_y \frac{\partial F}{\partial q}$$

In quasilinear case $\frac{\partial F}{\partial p}$ + $\frac{\partial F}{\partial q}$ were fns of $x, y, u$ alone, but now they depend on $p$ + $q$. So we need to track them using ODE's. Evidently (since $p = u_x$, $q = u_y$) we have

$$\frac{dp}{ds} = u_{xx}\dot{x} + u_{xy}\dot{y} = u_{xx}F_p + u_{xy}F_q$$

$$\frac{dq}{ds} = u_{xy}\dot{x} + u_{yy}\dot{y} = u_{xy}F_p + u_{yy}F_q$$

From $F(x, y, u, u_x, u_y) = 0$ we get

$$0 = F_x + F_u u_x + F_p u_{xx} + F_q u_{xy}$$

$$0 = F_y + F_u u_y + F_p u_{xy} + F_q u_{yy}$$

So

$$\frac{dp}{ds} = -F_x - F_u u_x = -F_x - F_u p$$

$$\frac{dq}{ds} = -F_y - F_u u_y = -F_y - F_u q$$

Thus we can recover soln of pde restricted to a suitable curve by solving the ODE system

$$\begin{aligned} \dot{x}(s) &= F_p \\ \dot{y}(s) &= F_q \\ \dot{u}(s) &= p F_p + q F_q \\ \dot{p}(s) &= -F_x - F_u p \\ \dot{q}(s) &= -F_y - F_u q \end{aligned}$$

(where RHS is evaluated at $(x(s), y(s), u(s), p(s), q(s))$ ).

To get a (local) soln $u(x,y)$ near a curve $S \subset \mathbb{R}^2$ we proceed as before: suppose $S$ is parametrized by $x = \gamma(\tau)$, $y = \mu(\tau)$ and we're given $u$ along $S$, i.e. $u(\gamma(\tau), \mu(\tau)) = g(\tau)$ is given. These clearly specify the initial data for $x(s), y(s), u(s)$ but what abt $p(s)$ + $q(s)$? Well, we know

$$(*)\begin{cases} \varphi_\tau = u_x \gamma_\tau + u_y \mu_\tau = p\gamma_\tau + q\mu_\tau \\ 0 = F(\gamma, \mu, \varphi, p, q). \end{cases}$$

In quasilinear case this was a linear system for $p+q$ (at $s=0$) + linear algebra told us whether it had a unique soln or not. In nonlinear case we must simply <u>assume</u> that

> (*) admits a unique soln for $p, q$, <u>and furthermore</u>
>
> $(\gamma_\tau, \mu_\tau)$ is not parallel to $(F_p, F_q)$

With this condn, ODE is solvable for each $\tau$ (for $s$ in nbhd of $0$).

As in quasilinear case: the fact that the $u(x,y)$ built up this way <u>does</u> solve the pde is an appln of Implicit Function Thm (see the books).

---

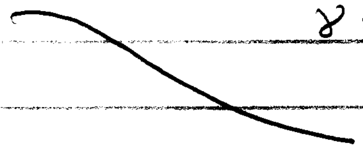
Now a word abt classification of $2^{\text{nd}}$ order linear pde. (Why? I want to introduce the <u>other</u> usage of the word "characteristic".) Consider

$$a(x,y)u_{xx} + 2b(x,y)u_{xy} + c(x,y)u_{yy} + d(x,y)u_x + e(x,y)u_y + f(x,y)u = g(x,y).$$

It is *elliptic* if $ac - b^2 > 0$
*hyperbolic* if $ac - b^2 < 0$
*parabolic* if $ac - b^2 = 0$

(Note that if $a, b, c$ are fns of $x, y$ then type can be different at different pts.)

*Question*: when can we specify $u$ + $\frac{\partial u}{\partial n}$ along a curve $\gamma$ in $(x,y)$ plane + thereby fully determine (at least formally) the function $u$ nearby?



$\gamma = (\gamma_1(s), \gamma_2(s))$

*Idea*: use eqn to solve for 2nd derivs along $\gamma$, if possible.

Know $\frac{du_x}{ds}$, $\frac{du_y}{ds}$ (since $u$ + $\frac{\partial u}{\partial n}$ determine the entire gradient $\nabla u$ along $\gamma$)

By chain rule

$$\frac{du_x}{ds} = u_{xx}\gamma_1'(s) + u_{xy}\gamma_2'(s)$$

$$\frac{du_y}{ds} = u_{xy}\gamma_1'(s) + u_{yy}\gamma_2'(s)$$

So we have 3 lin eqns in 3 unknowns $u_{xx}, u_{xy}, u_{yy}$ at each pt of $\gamma$:

$$\begin{aligned} a u_{xx} + 2b u_{xy} + c u_{yy} &= \text{known} \\ \gamma_1' u_{xx} + \gamma_2' u_{xy} \quad &= \text{known.} \\ \gamma_1' u_{xy} + \gamma_2' u_{yy} &= \text{known.} \end{aligned}$$

These determine $u_{xx}, u_{yy}, u_{xy}$ uniquely iff determinant $\neq 0$. But the det is

$$\det \begin{bmatrix} a & 2b & c \\ \gamma_1' & \gamma_2' & 0 \\ 0 & \gamma_1' & \gamma_2' \end{bmatrix} = a(\gamma_2')^2 + c(\gamma_1')^2 - 2b\gamma_1'\gamma_2'.$$

Note: normal to $\gamma$ is $(\gamma_2', -\gamma_1')$ so

$$\det \neq 0 \iff a n_1^2 + 2b n_1 n_2 + c n_2^2 \neq 0$$
where $n = \vec{n}_\gamma$

A curve where $\det \neq 0$ is called <u>noncharacteristic</u>. One where $\det = 0$ (ptwise) is called <u>characteristic</u>. Evidently:

elliptic $\iff$ every curve is noncharacteristic
hyperbolic $\iff$ two distinct families of characteristic curves
parabolic $\iff$ just one family of characteristic curves.

<u>Notes</u>:

(1) for wave eqn, the characteristic curves are of course the lines $x-t=$ const and $x+t=$ constant.

(2) for heat eqn $u_t - u_{xx} = 0$ the char curves are $t=$ constant (and indeed, we <u>cannot</u> specify both $u$ + $u_t$ independently at $t=0$!)

(3) extension of this discn to 2nd order pde in $\mathbb{R}^n$ leads to question: do $u$ + $\frac{\partial u}{\partial n}$ along a <u>codim 1 surface</u> $S$ permit one to formally solve for all 2nd derivs of $u$ along $S$? If eqn is $\sum a_{ij} \frac{\partial^2 u}{\partial x_i \partial x_j} + \text{lot} = 0$ then answer is yes ($D^2 u$ is formally determined) iff $\sum a_{ij} n_i n_j \neq 0$, where $\vec{n} = \vec{n}_S$ is normal vector to $S$. We say $S$ is a <u>noncharacteristic surface</u>

Evidently: for 1st order eqns, method of characteristics leads to curves (in $(x,y)$ space for lin eqns, but in $(x,y,u,p,q)$ space for nonlin case). But for 2nd order eqns, preceding discn leads to <u>codim-one surfaces</u> being characteristic or noncharacteristic.

Q: When $S$ is noncharacteristic, so $D^2u$ is formally determined, what abt higher derivs?
Ans: They're all formally determined.

Q: Does this mean there's actually a local soln $u$? Ans: no, not in general. But yes if data + $S$ are analytic (This is the Cauchy-Kovalevskaya Theorem)