PDE – Lecture 11, 11/25/2014

New topic today: nonlinear conservation laws, emphasizing scalar conservation laws (since we can understand them more or less completely), including shock waves and admissibility conditions.

In our texts the parts assoc this topic are

- Guenther + Lee

  §1.7 (ideal gas as a cons law, and derivation of linear wave eqn as descr of acoustic waves)

  §12.3 (concise discussion of scalar conservation laws, including shocks + admissibility + a basic uniqueness theorem)

- Evans

  §3.4 (excellent treatment of scalar cons laws; it couples to his treatment in §3.3 of Hamilton-Jacobi, which we'll do afterward; includes results on uniqueness + decay that we won't have time to do)

Motivation: many physical problems lead to nonlinear, 1st order pde's taking the form of conservation laws. We'll eventually focus on examples where "space" is one-dimensional, but first lets discuss a multidim'l example:

Flow of an ideal gas (cf Guenther + Lee §1,7)

$$(*) \begin{cases} \rho_t + \operatorname{div}(\rho \vec{v}) = 0 \\ (\rho v_i)_t + \operatorname{div}(\rho v_i \vec{v}) = -\nabla_i p \qquad i=1,2,3 \end{cases}$$

where $\rho(x,t)$ = density
$\vec{v}(x,t) = (v_1, v_2, v_3)$ = velocity
$p(x,t)$ = pressure

and the system is closed by a pressure-density law such as

$$\frac{p}{p_0} = \left(\frac{\rho}{\rho_0}\right)^{\lambda}$$

The pde's reflect cons of mass + cons of linear momentum, eg

cons of mass: $\frac{d}{dt}\int_D \rho \, dx = -\int_{\partial D} \rho \vec{v}\cdot n \, dA$ for any region D

true for all $D \iff \rho_t + \mathrm{div}(\rho \vec{v}) = 0$.

Brief digression: derivation of acoustic wave eqn by linearization. If $\rho/\rho_0 = 1+u$ with u small, and if $\vec{v}$ is also small, then

$$(\rho/\rho_0)^\lambda = (1+u)^\lambda \simeq 1 + \lambda u$$

$$\Rightarrow [\rho_0(1+u)]_t + \mathrm{div}[\rho_0(1+u)v] = 0 \quad \text{linearizes}$$

$$\text{to} \quad u_t + \mathrm{div}\, v = 0$$

and $(\rho_0(1+u)v_i)_t + \mathrm{div}(v_i \vec{v} \rho) + \nabla_i(p_0(1+u)^\lambda)$

$$\text{linearizes to} \quad \rho_0 v_t + p_0 \lambda \nabla u = 0$$

Substitution $\Rightarrow u_{tt} - \frac{p_0}{\rho_0}\lambda \Delta u = 0$,

the linear wave eqn with wavespeed $\left(\frac{p_0}{\rho_0}\lambda\right)^{1/2}$.

But: such a linearization is not always justified. (For example: sonic booms are not described by a linear wave eqn.)

Our goal today is to consider a more fully nonlinear setting, focusing on problems where "space is one dimensional" (since they are more accessible than 3D, and they still include some interesting examples). We therefore turn to

Scalar conservation laws

$$u_t + (F(u))_x = 0 \qquad t > 0$$

$$u = g(x) \qquad \text{at } t = 0$$

where u is scalar-valued and $x \in \mathbb{R}$.

Example 1: Burger's eqn $\quad u_t + \frac{1}{2}(u^2)_x = 0$

For smooth solns, eqn is equiv to $u_t + u u_x = 0$. Describes "Newtonian motion of noninteracting particles in 1D" since if

$z(x_0, t)$ = posn at time t of particle originally at x_0

$u(x, t)$ = velocity of particle that's at position x at time t

Then

$$\frac{\partial z}{\partial t} = u(z(x_0, t), t) \qquad \text{by defn};$$

and Newton's law says

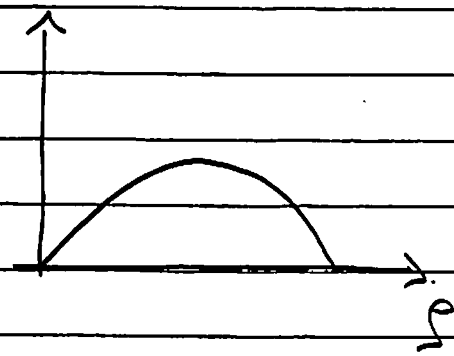
$$\frac{\partial^2 z}{\partial t^2} = 0 \quad \text{ie} \quad u_t + u u_x = 0$$

Example 2: Traffic flow $\rho_t + [Q(\rho)]_x = 0$
where

$\rho(x,t)$: traffic density $\left(\frac{\text{cars}}{\text{meter}}\right)$ at x,t

$Q(\rho)$ = rate of traffic flow $\left(\frac{\text{cars}}{\text{hour}}\right)$ at x,t

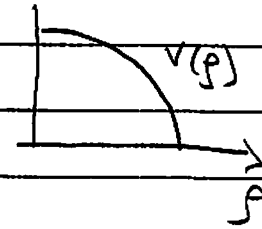
The eqn expresses "cons of cars". Typical form of Q



since $\rho = 0 \Rightarrow$ no cars $\Rightarrow Q = 0$

and at maximal ρ, bumper to bumper traffic $\Rightarrow$ velocity of each car must be small to maintain proper following distance $\Rightarrow Q(\rho) \to 0$.

[implicit here: $Q(\rho) = \rho V(\rho)$ where $V(\rho)$ = velocity of cars at density ρ



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Detailed study of traffic flow by this approach: Lighthill + Whitham, Proc Roy Soc London

Ser A vol 229 (1955) 317-345

Example 3: Flood waves in a river

$A(x,t)$ = cross-sectional area of flow
(vol of water per unit length)

$q(x,t)$ = flux of water
(vol of water per unit time)

Shape of riverbed (+ physical effects such as viscosity, + overall slope [assumed constant for simplicity]) gives $q = q(A)$.

"Cons of water" gives (assuming no sources or sinks)

$$\frac{d}{dt}\int_a^b A\,dx = -q(b) + q(a)$$

True for all $a, b \Rightarrow A_t + q_x = 0$.

Detailed study of flood waves by this viewpt: Lighthill + Whitham, Proc Roy Soc London Ser A vol 229 (1955) 281-316

OK, so let's think now abt solns of a scalar cons law $u_t + (F(u))_x = 0$. Initial tasks are to explain

- repn of smooth solns by "method of characteristics"
- formation of shocks
- Rankine - Hugoniot condn (sets the speed of a shock)
- admissibility condition (required, for uniqueness)

<u>Representation of smooth solns by method of characteristics.</u>

Method of characteristics is a way of 1st order pde's to ODE's; specialization to present setting is especially simple. If u is C^1 we can write eqn as

$$u_t + c(u)\, u_x = 0$$

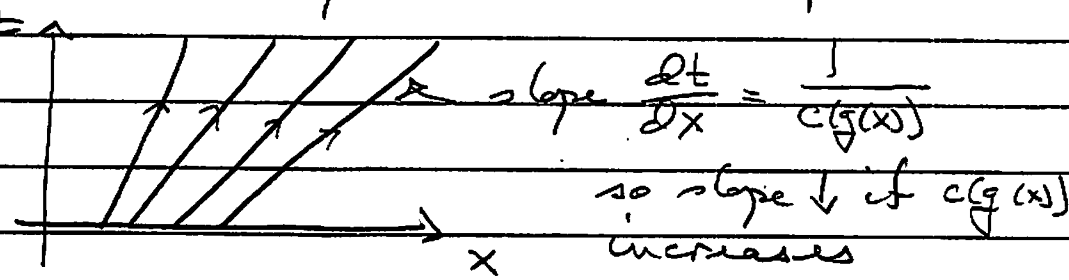
where $c(u) = F'(u)$. Evidently, <u>along space-time curve</u> <u>$dx/dt = c(u)$</u> we have

$$\frac{d}{dt} u(x(t), t) = u_x \dot{x} + u_t = 0$$

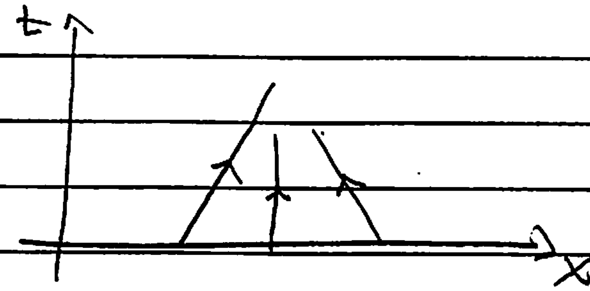
ie <u>u = const along this curve</u> (which is in fact a <u>line</u> in this setting).

Shocks may form or not, depending on initial data:

- If $c(g(x))$ increases as x increases then the characteristics spread out; no problem here.



- But if $c(g(x))$ decreases as x increases then characteristics must eventually cross. So soln cannot be C^1 (it develops a shock)



When, exactly, does C^1 soln break down? Rephrase prev descn by

$$u = g(\xi) \text{ when } x = \xi + t\,c(g(\xi))$$

Observe that $\frac{dx}{d\xi} = 1 + t\,\mathcal{C}'(\xi)$ where $\mathcal{C}(\xi) = c(g(\xi))$

So the map $\xi \to x(t,\xi)$ is single-valued provided

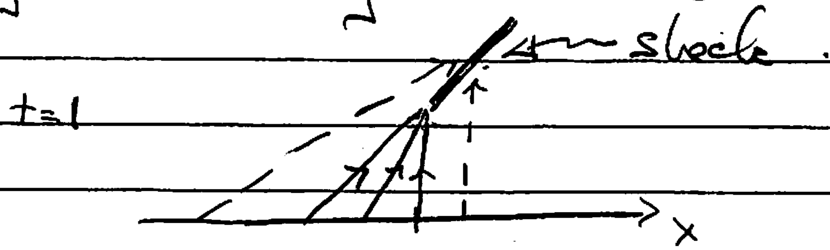
$1+t\varphi'(\xi) > 0$. Breakdown time is thus

$$t_* = \frac{-1}{\varphi'(\xi_*)}$$ where ξ_* achieves largest $|\varphi'(\xi)|$ among all ξ st $\varphi'(\xi) < 0$.

<u>An explicit example</u>: for Burgers eqn
$u_t + \frac{1}{2}(u)^2_x = u_t + u u_x = 0$, if initially

$$u(x,0) = \begin{cases} 1 & x \le 0 \\ 1-x & 0 \le x \le 1 \\ 0 & x \ge 1 \end{cases}$$

soln is singular at time 1 (this example is non-generic: many char's coalesce at time 1).



So, we want to consider <u>solns with shocks</u>, ie piecewise C^1 solns.

Speed of shock is set by "<u>Rankine-Hugoniot condition</u>," which expresses cond that cons law holds "weakly" across the shock.

Specifically: for $u_t + (F(u))_x = 0$, for a piecewise C^1 (but disconts) soln across a shock at $x = s(t)$, RH cond says

$$\frac{[F(u)]}{[u]} = \frac{ds}{dt}$$

where $[u]$ = jump in u.

1st explanation: return to integrated form of law. It says, for $a < s(t) < b$,

$$\frac{d}{dt}\int_a^b u + F(u)(b) - F(u)(a) = 0$$

which we write

$$\frac{d}{dt}\left(\int_a^{s(t)} u + \int_{s(t)}^b u\right) + F(b) - F(u_R(s(t))) + [F(u)] + F(u_L(s(t))) - F(a).$$

If $u_t + (F(u))_x$ holds on both sides of the shock, then we easily deduce

$$s'(t)\left(u(s(t))_L - u(s(t))_R\right) + [F(u)] = 0$$

i.e. $$s'(t) = \frac{[F(u)]}{[u]}.$$

<u>2nd explanation:</u> a "weak solution" should satisfy

$$\iint u\,\varphi_t + F(u)\,\varphi_x = 0$$

for any compactly supported $\varphi = \varphi(x,t)$.
Equivalently: the vector field $(u, F(u))$ is "weakly divergence free" in (t,x) space.

<u>Lemma:</u> in $\mathbb{R}^n$, suppose a vector field $\vec{\xi}(x)$ is piecewise smooth but discontinuous across a surface S. Then ξ is "wkly div free" in the sense that

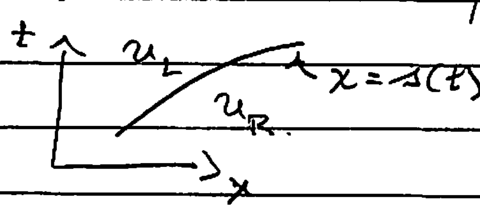
$$\int (\operatorname{div} \xi)\,\varphi =_{\text{defn}} -\int \langle \xi, \nabla\varphi \rangle = 0$$

for all cptly sptd φ iff

- $\operatorname{div} \xi = 0$ in each region where it is smooth, and
- $[\xi \cdot n] = 0$ across S

(Sketch of proof: apply Green's formula separately to the regions on both sides of S.)

Apply to cons law:



use $\xi = (F(u), u)$ in $\mathbb{R}^2 = (x,t)$;
normal is $\vec{n} = (1, -\dot{s}) / (1+\dot{s}^2)^{1/2}$ so
cond $[\xi \cdot n] = 0$ becomes

$$F(u_L) - \dot{s} u_L = F(u_R) - \dot{s} u_R$$

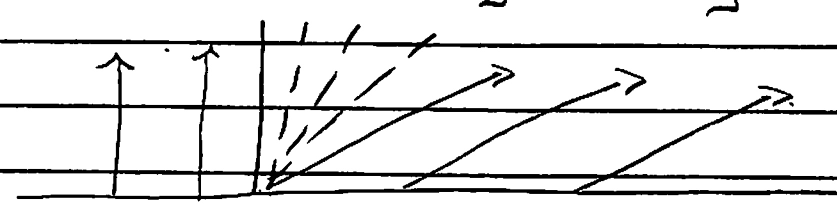
i.e. $$\dot{s} = \frac{F(u_R) - F(u_L)}{u_R - u_L}$$ as expected.

But we're not done, because without any further conds wk solns are not unique.

Example: For Burgers' eqn $u_t + \frac{1}{2}(u^2)_x = 0$ with piecewise constant initial data

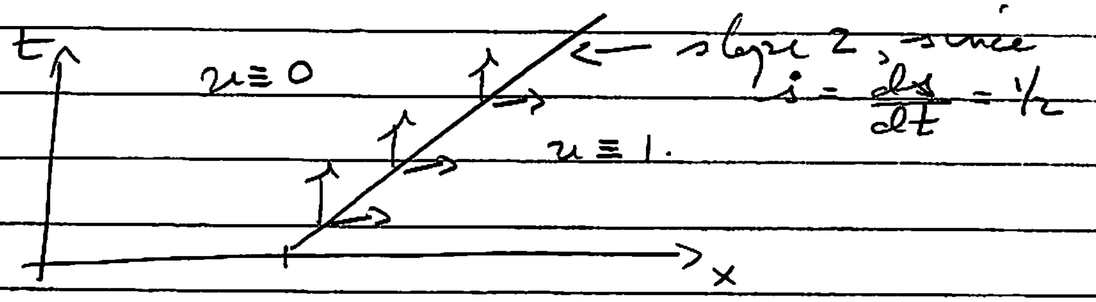
$$u(x,0) = \begin{cases} 0 & x < 0 \\ 1 & x > 0 \end{cases}$$

we can take u to be given by a "fan"



$u \equiv 0$ for $x < 0$, $u = \frac{x}{t}$ for $0 < x < t$, $u \equiv 1$ for $x \geq t$

(this is the "admissible soln", to be explained presently) or we could take u to have a shock



(this is a "non-admissible shock", to be explained below; note that for Burgers, RH says $\dot{s} = \frac{\frac{1}{2}(u_R^2 - u_L^2)}{u_R - u_L} = \frac{u_R + u_L}{2}$.)

The physical origin of the <u>admissibility condition</u> which restores uniqueness is that we want $u = \lim_{\varepsilon \to 0} u_\varepsilon(x,t)$ where

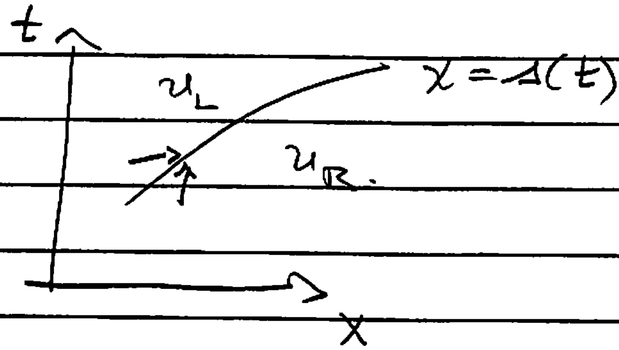
$$\partial_t u_\varepsilon + \partial_x \big(F(u_\varepsilon)\big) = \varepsilon \partial_{xx} u_\varepsilon$$

(RHS = effect of a small viscous dissipation previously excluded from the model).
This seems at first hard to use, but if F is strictly convex (F' is strictly increasing) it is equivalent to a very simple condition:

<u>version 1</u>: $F'(u_L) > \frac{ds}{dt} > F'(u_R)$

<u>version 2: No characteristics should originate from shocks</u>

Equivalence of versions (1) + (2) is seen in picture:



char on left has dirn $(F'(u_L), 1)$
" " right " " $(F'(u_R), 1)$.
shock has dirn $(s'(t), 1)$

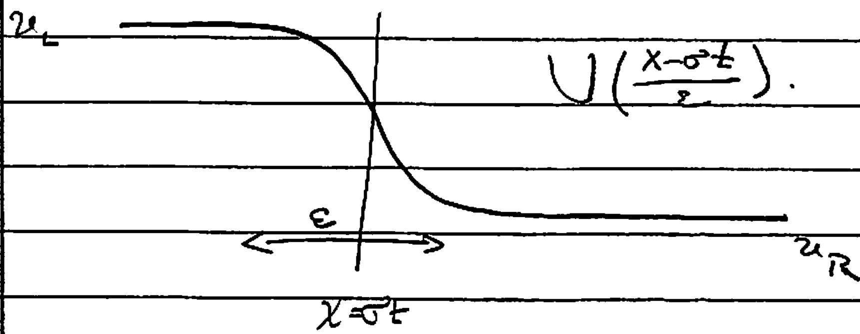
To explain the form of the admissibility condition, let's assume that a piecewise constant solution with a shock at $x = \sigma t$ (shock speed σ) is the limiting behavior as $\varepsilon \to 0$ of a travelling wave solution

$$u_\varepsilon(x,t) = U\left(\frac{x - \sigma t}{\varepsilon}\right)$$

The "shock profile" $U(\xi)$ should then solve

$$-\sigma U' + (F(U))' = U'' \qquad -\infty < \xi < \infty$$

with $U \to u_L$ as $\xi \to -\infty$ and $U \to u_R$ as $\xi \to +\infty$,



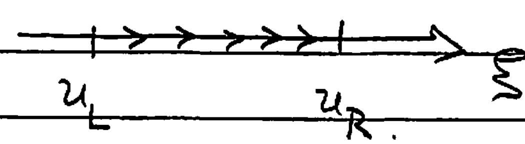
Assuming existence of such U, integration gives

$$-\sigma U + F(U) = U' + c$$

Evidently $c = -\sigma U_L + F(U_L) = -\sigma U_R + F(U_R)$ (since by hypoth $U' \to 0$ at $\pm\infty$) which forces

$$\sigma = \frac{[F(U)]}{[U]}$$

as expected. But we also get more: The ODE $U' = F(U) - \sigma U - c$ has u_L and u_R as crit pts, and (by hypothesis) a soln that goes from one to the other



So the linearization should be unstable at u_L and stable at u_R. This gives

$$F'(u_L) - \sigma \geq 0, \quad F'(u_R) - \sigma \leq 0.$$

This is (modulo strictness of the inequalities) the same as our admissibility condn.

Where to go from here? Natural topics are

A) Uniqueness of admissible solns.
(More precisely: if u + v are piecewise smooth solns with admissible shocks + same initial data then $u = v$).

B) The "Lax-Oleinik solution formula", which gives an (almost explicit) representation of the (unique) admissible soln.

C) The "Hopf-Cole transf", which reduces the viscous Burgers' eqn $u_t + \frac{1}{2}(u^2)_x = \varepsilon u_{xx}$ to a linear heat eqn, permitting the $\varepsilon \to 0$ limit to be extracted using "Laplace asymptotics".

Topic (A) is interesting + well worth reading about - see §12.3 of Guenther + Lee.

Topic (B) will come naturally with our treatment of Hamilton-Jacobi eqns

Topic (C) is interesting because it reveals how it is that small viscosity selects a particular soln, at least for Burgers' eqn. So let's discuss it in some detail. Recall: our goal is to understand limiting behavior as $\varepsilon \to 0$ of $u = u_\varepsilon$ that solves

$$\begin{aligned} u_t + u u_x &= \varepsilon u_{xx} && t > 0 \\ u &= g && \text{at } t = 0, \end{aligned}$$

Step 1 Integrate once in x, to get

$$\begin{aligned} w_t + \tfrac{1}{2} w_x^2 &= \varepsilon w_{xx} && t > 0 \\ w &= h && t = 0 \end{aligned}$$

where $w_x = u$, $h_x = g$.

Step 2 Reduce to linear heat eqn by ansatz $v = \varphi(w)$. To choose φ, note that if $v = \varphi(w)$ then

$$v_t = \varphi'(w)\, w_t \qquad v_x = \varphi'(w)\, w_x$$

$$v_{xx} = \varphi''(w)\, w_x^2 + \varphi'(w)\, w_{xx}$$

so

$$V_t - \varepsilon V_{xx} = \varphi'(w)\, w_t - \varepsilon \varphi''(w)\, w_x^2 - \varepsilon \varphi'(w)\, w_{xx}$$

$$= \varphi'(w)\left[\cancel{\varepsilon w_{xx}} - \tfrac{1}{2} w_x^2\right] - \varepsilon \varphi''(w)\, w_x^2$$
$$\cancel{- \varepsilon \varphi'(w)\, w_{xx}}$$

For RHS to vanish we should choose φ s.t.

$$\varepsilon \varphi'' + \tfrac{1}{2}\varphi' = 0$$

$$\Rightarrow \text{ use } \varphi(\xi) = e^{-\xi/2\varepsilon}$$

Thus: $V = e^{-w/2\varepsilon}$ solves the linear heat eqn $V_t - \varepsilon V_{xx} = 0$, with initial data $V(x,0) = e^{-h/2\varepsilon}$.

Outcome of step 2:

$$u(x,t) = \frac{\partial}{\partial x}\left[-2\varepsilon \log V(x,t)\right], \text{ where}$$

$$V(x,t) = \frac{1}{2\sqrt{\pi \varepsilon t}} \int_{-\infty}^{+\infty} e^{-(x-y)^2/4\varepsilon t}\, e^{-h(y)/2\varepsilon}\, dy$$

or equivalently

$$u(x,t) = \frac{\int_{-\infty}^{+\infty} \frac{x-y}{t}\, e^{-K(x,y;t)/2\varepsilon}\, dy}{\int_{-\infty}^{+\infty} e^{-K(x,y;t)/\varepsilon}\, dy}$$

where $$K(x,y;t) = \frac{|x-y|^2}{2t} + h(y)$$

Step 3: Apply "Laplace asymptotics" i.e. the following lemma (see eg Evans §4.5.2): if $k(y)$ + $l(y)$ are cont's, k grows at least quadratically + l grows at most linearly, and if $\exists$ unique y_0 st

$$k(y_0) = \min_{y \in \mathbb{R}} k(y)$$

Then
$$\lim_{\varepsilon \to 0} \frac{\int_{-\infty}^{+\infty} l(y) e^{-k(y)/\varepsilon} dy}{\int_{-\infty}^{+\infty} e^{-k(y)/\varepsilon} dy} = l(y_0)$$

Conclusion:

(*) $$\lim_{\varepsilon \to 0} u^\varepsilon(x,t) = \frac{x - y_0(x,t)}{t}$$

where
$$y_0(x,t) = \operatorname{argmin}_y \left\{ \frac{|x-y|^2}{2t} + h(y) \right\}$$

[Note: we'll show when we get to Hamilton-Jacobi eqns that the limiting behavior of w_ε is

(**) $$w(x,t) = \min_{y \in \mathbb{R}} \left\{ \frac{|x-y|^2}{2t} + h(y) \right\}$$

which is of course consistent with (*).]