PDE - Lecture 10, 11/18/2014

Today: linear wave eqn in $\mathbb{R}^2$ + $\mathbb{R}^3$; also just a bit about numerical schemes, and about $u_{tt} - \Delta u$ = nonzero source term.

Focus first on 3D wave eqn

$$u_{tt} - \Delta u = 0 \quad \text{in } \mathbb{R}^3$$

$$\left.\begin{array}{l} u = f \\ u_t = g \end{array}\right\} \text{ at } t = 0$$

There's a formula that's almost as simple as the 1D case:

$$u(x,t) = \fint_{|y-x|=t} \left[f(y) + t\, g(y) + (y-x)\cdot\nabla f(y) \right] dy$$

(*)

$$= \frac{1}{4\pi t^2} \int_{|y-x|=t} \left[\text{———} \right] dy$$

This can also be written

$$(**) \qquad u(x,t) = t \fint_{|y-x|=t} g\, dy + \frac{d}{dt}\left[t \fint_{|y-x|=t} f\, dy \right].$$

(To check the equivalence: the not-completely-obvious part is that

$$\frac{d}{dt} \int_{|y-x|=t} f \, dy = \frac{1}{t} \int_{|y-x|=t} (y-x)\cdot\nabla f \, dy .$$

In fact the LHS is

$$\frac{d}{dt} \int_{|z|=1} f(x+tz)\, dz = \int_{|z|=1} z\cdot\nabla f(x+tz)\, dz$$

$$= \int_{|y-x|=t} \frac{y-x}{t}\cdot\nabla f(y)\, dy$$

which gives the desired result.)

This soln formula can be obtained

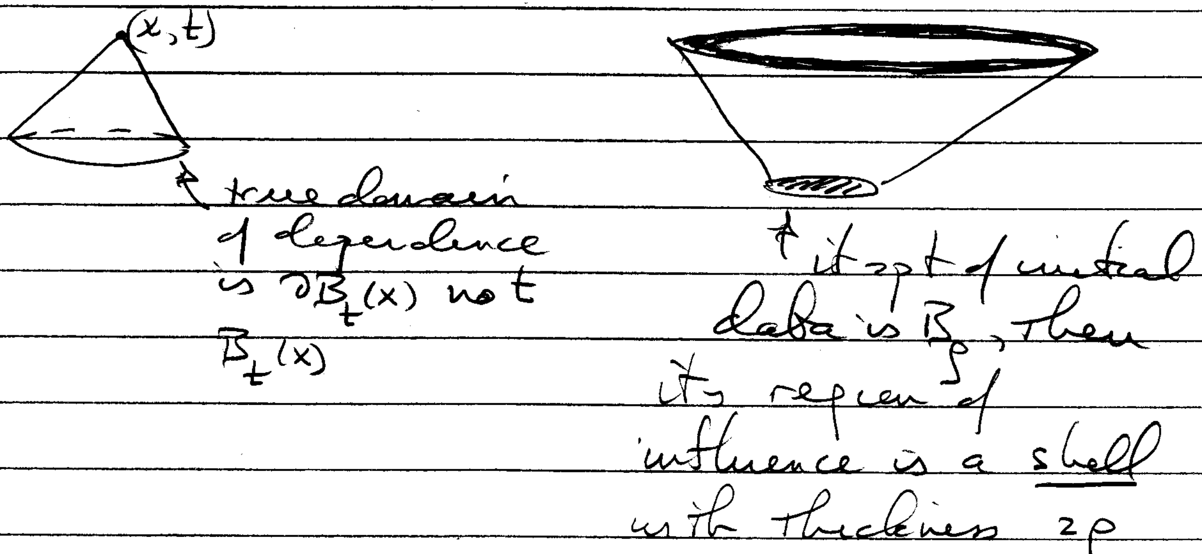
- by a process similar to what we did for the heat eqn, using something similar to a "fundamental soln" (see Guenther + Lee § 10.4 for this)

or

- by the "method of spherical means" (see Evans § 2.4 or F. John's book § 5.1 for this; I'll sketch it below)

Accepting the formula for now, let's observe some key properties:

a) in $\mathbb{R}^3$, true domain of dependence is a sphere not a ball: $u(x,t)$ depends only on data $(g, f, \nabla f)$ on y st $|y-x| = t$. (This is special to wave eqn in odd dimensions $n = 3, 5, 7$, etc; note that $n=1$ is different.)



b) in $\mathbb{R}^3$, if initial data are compactly supported then $\|u\|_{L^\infty(\mathbb{R}^3)} \to 0$ like $1/t$ as $t \to \infty$

Intuition: energy is conserved ($\int u_t^2 + |\nabla u|^2 dx$ is const) but support spreads (it lives in a growing shell) so amplitude should decay.

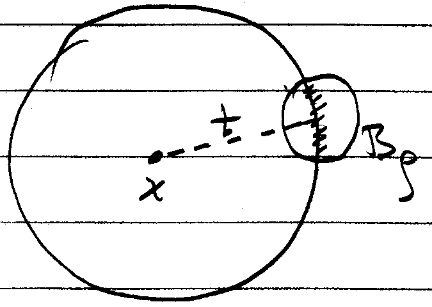
justification: using version (*) of the formula, suppose $f + g$ are supported in B_ρ. At time t and for given x, the integrand in

$$u = \fint_{|y-x|=t} \left[f + tg + (y-x)\cdot\nabla f \right]$$

vanishes except on

$$\{ y : |y-x| = t \ \& \ y \in B_\rho \}$$

which has 2D area $\sim \rho^2$ if t is large



and on its spt the integrand is of order t. So u is at most

$$\underbrace{\frac{\text{const}}{t^2}}_{\text{normalization}} * \overset{\text{integrand}}{t} * \underset{\text{area}}{\rho^2} \sim \frac{C\rho^2}{t}.$$

[Note that 1D wave eqn is very different: solns with compactly supported initial data do not decay as $t \to \infty$.]

c) Soln of 3D wave eqn is in general <u>less regular</u> than data (since soln formula involves ∇f as well as f)

d) Essential features of soln are evident even in radial setting, since formula tells us that

$$\underbrace{u(0,t) = t \int_{|y|=t} g + \frac{d}{dt}\left[t \int_{|y|=t} f \right]}_{\text{feels only the averages of } f \text{ and } g \text{ on spheres of various radii}}$$

Soln to 2D wave eqn

$$u_{tt} - \Delta u = 0 \quad \text{in } \mathbb{R}^2$$

$$\left.\begin{array}{l} u = f \\ u_t = g \end{array}\right\} \text{ at } t=0$$

can be deduced from the 3D formula (and a similar calcn can be done for any even dim $n = 2k$, starting from the soln formula in dim $2k+1$). The idea, known as the "method of descent," is simple (at least conceptually): extend the initial data to $\mathbb{R}^3$, as fns indep of x_3; then solve 3D wave eqn; then evaluate soln at $(x_1, x_2, 0; t)$.

What comes out is

$$(***) \quad u(x_1, x_2; t) = \frac{1}{2\pi} \int_{|y-x|<t} \frac{g(y)}{(t^2 - |y-x|^2)^{1/2}} \, dy_1 \, dy_2$$

$$+ \frac{d}{dt} \left[\frac{1}{2\pi} \int_{|y-x|<t} \frac{f(y)}{(t^2 - |y-x|^2)^{1/2}} \, dy_1 \, dy_2 \right]$$

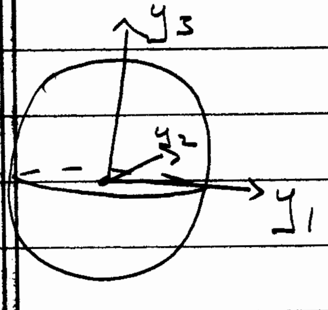
<u>A key property of the 2D soln formula</u>: in 2D the domain of dependence is the entire ball $|y-x| \le t$. So: in 2D a localized disturbance takes the expected time to reach the observer (dictated by the wave-speed) but thereafter the source will "continue to be seen forever" (albeit with decaying amplitude).

<u>Derivation of (***)</u>: we start from our 2nd version of the 3D formula: $u = t \fint_{|y-x|=t} g + \frac{d}{dt}\left[t \fint_{|y-x|=t} f \right]$

The two terms are similar, so it suffices to focus on the one involving g. The point is that

$$\frac{1}{4\pi t^2}\int_{|y_1-x_1|^2+|y_2-x_2|^2+y_3^2=t^2} g(y_1,y_2)\ d(\text{area})$$

can be written as an integral wrt (y_1,y_2):



center $=(x_1,x_2,0)$

$$d(\text{area})=\left[1+\left(\frac{\partial y_3}{\partial y_1}\right)^2+\left(\frac{\partial y_3}{\partial y_2}\right)^2\right]^{1/2}dy_1\,dy_2$$

$$=\frac{t}{|y_3|}\,dy_1\,dy_2$$

The prefactors are $\frac{1}{2\pi}$ not $\frac{1}{4\pi}$ in the 2D formula since the upper + lower hemispheres both contribute equally in the preceding calculation.

<u>What if RHS is nonzero</u>? In other words what about

$$u_{tt}-\Delta u=W(x,t)\quad \text{in } \mathbb{R}^n,\ \text{for } t>0$$

$$\left.\begin{array}{l} u=f \\ u_t=g \end{array}\right\}\ \text{at } t=0$$

(This discn is <u>independent</u> of spatial dimension.) Like the heat eqn, we can view wave eqn as an "ODE in function space"; in fact

$$u_{tt} = \Delta u \quad \text{iff} \quad \frac{d}{dt}\begin{pmatrix} u \\ u_t \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ \Delta & 0 \end{pmatrix}\begin{pmatrix} u \\ u_t \end{pmatrix}.$$

So we can use something like the ODE "variation of parameters formula" to represent a solution with nonzero w.

But the "ODE framework" is perhaps more confusing than useful. So instead let's <u>skip straight to the answer</u> that comes out: to solve $u_{tt} - \Delta u = w(x,t)$ we look for

$$u = u_1 + u_2$$

where

u_1 solves the pbm with no source term, and the given initial data

u_2 solves the wave eqn with source term w and initial data 0 ($\partial_t u_2 = 0$ and $u_2 = 0$ at $t=0$).

We understand u_1; what abt u_2? Ans is:

$$u_2(x,t) = \int_0^t U(x,t;s)\,ds$$

where for fixed s, $U(x,t;s)=0$ for $t<s$, and it solves pde

$$U_{tt} - \Delta U = 0 \qquad t>s$$

$$\left.\begin{array}{l} U(x,s;s)=0 \\ U_t(x,s;s)=w(x,s) \end{array}\right\} \text{"initial conds" at } t=s$$

We can check by direct calcn that this works:

$$\partial_t u_2 = \cancel{U(x,t,t)}^{\,0} + \int_0^t U_t(x,t;s)\,ds$$

$$\partial_{tt} u_2 = U_t(x,t;t) + \int_0^t U_{tt}(x,t;s)\,ds$$

$$= w(x,t) + \int_0^t \Delta U(x,t;s)\,ds.$$

$$\Delta u_2 = \int_0^t \Delta U(x,t;s)\,ds$$

So $(\partial_{tt} - \Delta) u_2 = w$. Also $u_2(x,0)=0$ and $\partial_t u_2(x,0)=0$, as desired.

What about numerical solutions? Simplest approach – one that's appropriate even for nonlinear wave eqns – is to discretize space but keep time cont's. Discrete scheme should maintain the "Hamiltonian structure", ie should have

$$\frac{d}{dt}\,(\text{kinetic} + \text{potential energy}) = 0,$$

for a suitably discretized version of the "potential energy". For example, in 1D on $[0,L]$ with $u=0$ at endpts $x=0,L$, consider nonlin wave eqn

$$u_{tt} - u_{xx} + u^3 = 0.$$

Assoc cons of energy is

$$\frac{d}{dt}\left(\int_0^L \tfrac{1}{2}u_t^2 + \tfrac{1}{2}u_x^2 + \tfrac{1}{4}u^4\,dx\right) = 0$$

Discretizing in space, we should solve for

$$u_j(t) \simeq u(j\Delta x, t) \quad \text{where } \Delta x = \frac{L}{N}$$

We then want

$$\frac{d}{dt}\left(\sum \tfrac{1}{2}\dot{u}_j^2 + \frac{|u_j - u_{j-1}|^2}{2(\Delta x)^2} + \tfrac{1}{4}u_j^4\right) = 0$$

which we can achieve by solving

$$\ddot{u}_j = \frac{u_{j-1} + u_{j+1} - 2u_j}{(\Delta x)^2} - u_j^3$$

(The bc $u_0 = 0$ and $u_N = 0$ are needed to get $\ddot{u}_1$ and $\ddot{u}_N$; the ode needs initial conds for $u_j(0)$ and $\dot{u}_j(0)$ as expected.)

What about a time-discrete scheme? The simplest (explicit) method for the 1D linear wave eqn is

$$\frac{u_j(t+\Delta t) + u_j(t-\Delta t) - 2u_j(t)}{(\Delta t)^2} = \frac{u_{j+1}(t) + u_{j-1}(t) - 2u_j(t)}{(\Delta x)^2}$$

which rearranges to

$$u_j(t+\Delta t) = -u_j(t-\Delta t) + \left(\frac{\Delta t}{\Delta x}\right)^2 \left[u_{j+1}(t) + u_{j-1}(t)\right] + 2\left(1 - \left(\frac{\Delta t}{\Delta x}\right)^2\right) u_j(t)$$

Fact: this scheme is unstable if $(\Delta t/\Delta x)^2 > 1$, but it is stable if $(\Delta t/\Delta x)^2 < 1$. (See eg Strauss §8.3.) Heuristic interpretation: in this discrete scheme info can only travel one unit Δx at each time step Δt, ie propn speed is $\Delta x/\Delta t$. So surely we need $\Delta x/\Delta t > 1$ as a necessary condn for the scheme to get the soln (approximately) correct.

The rest of these notes derive the 3D soln formula using the method of spherical means (my notation follows F. John's treatment, but the one in Evans is fundamentally the same).

<u>Notation</u>: if $\varphi = \varphi(x)$ is defined on $\mathbb{R}^n$ then

$$M_\varphi(x,r) = \text{avg of } \varphi \text{ on } \underline{\text{sphere}} \text{ of rad } r \text{ centered at } x$$

$$= \fint_{|y-x|=r} \varphi(y)\,dy = \frac{1}{n\alpha(n)r^{n-1}} \int_{\partial B_r(x)} \varphi$$

[Note: if $\alpha(n)r^n = $ vol of B_r in $\mathbb{R}^n$, then $n\alpha(n)r^{n-1}$ is area of ∂B_r]. We take the convention that

$$M_\varphi(x,r) = M_\varphi(x,-r) \quad \text{for } r<0,$$

so that

$$M_\varphi(r) = \frac{1}{n\alpha(n)} \int_{|z|=1} \varphi(x+rz)\,dz$$

holds even for $r<0$.

<u>Claim</u>: If u solves wave eqn in $\mathbb{R}^n$, then for any fixed x, M_u solves the pde in $r+t$:

$$\partial_{rr} M_u + \frac{n-1}{r}\partial_r M_u = \partial_{tt} M_u .$$

[Note: This is just the wave eqn in $\mathbb{R}^n$ in radial coords, since when $\varphi = \varphi(r)$ in $\mathbb{R}^n$ we have $\Delta\varphi = \varphi_{rr} + \frac{n-1}{r}\varphi_r$. A painless way to see this is to consider the 1st variation of $\int |\nabla\varphi|^2 dx = c\int |\varphi'(r)|^2 r^{n-1} dr$.]

<u>Proof of the claim, by direct calculation</u>:

<u>Observe first that</u> for any φ,

$$\partial_r M_\varphi = \frac{r}{n} \fint_{B_r(x)} \Delta\varphi$$

since

$$\partial_r M_\varphi(x) = \frac{1}{n\alpha(n)} \partial_r \int_{|z|=1} \varphi(x+rz)\, dz$$

$$= \frac{1}{n\alpha(n) r^{n-1}} \int_{\partial B_r(x)} \frac{\partial\varphi}{\partial n}$$

$$= \frac{1}{n\alpha(n) r^{n-1}} \int_{B_r(x)} \Delta\varphi$$

$$= \frac{r}{n} \frac{1}{\alpha(n) r^n} \int_{B_r(x)} \Delta\varphi$$

[Digressing for a moment: preceding observation shows that if φ is C^2 then $\partial_r M_\varphi \to 0$ as $r \to 0$. Also, differentiating in r,

$$\partial_{rr} M_\varphi = -(n-1) r^{-n} \cdot \frac{1}{n\alpha(n)} \int_{B_r(x)} \Delta\varphi$$

$$+ \frac{1}{n\alpha(n) r^{n-1}} \int_{\partial B_r(x)} \Delta\varphi.$$

$$= -\left(\frac{n-1}{n}\right) \fint_{B_r(x)} \Delta\varphi + \fint_{\partial B_r(x)} \Delta\varphi.$$

so $\lim_{r\to 0} \partial_{rr} M_\varphi = \frac{1}{n}\Delta\varphi(x)$ whenever φ is C^2.]

Continuing now the proof of the claim: if u solves the wave eqn we have

$$\partial_r M_u = \frac{1}{n\alpha(n) r^{n-1}} \int_{B_r(x)} u_{tt}$$

so $$r^{n-1} \partial_r M_u = \frac{1}{n\alpha(n)} \int_0^r d\rho \int_{|y-x|=\rho} u_{tt}$$

whence

$$\partial_r\left(r^{n-1}\partial_r M_u\right) = \frac{1}{n\alpha(n)}\int_{|y-x|=r} u_{tt}$$

$$= r^{n-1}\partial_{tt} M_u$$

Thus $\partial_{tt} M_u = \frac{1}{r^{n-1}}\partial_r\left(r^{n-1}\partial_r M_u\right)$,

which is equivalent to the claim.

Now we deduce the 3D soln formula. When $n=3$ our claim says

$$(rM_u)_{tt} = (rM_u)_{rr}$$

so rM_u solves the 1D wave eqn in $r + t$. By 1D soln formula,

$$rM_u(x, r; t) = \tfrac{1}{2}\left[(r+t)M_f(x; r+t) + (r-t)M_f(x; r-t)\right] + \tfrac{1}{2}\int_{r-t}^{r+t} \xi M_g(x,\xi)\, d\xi$$

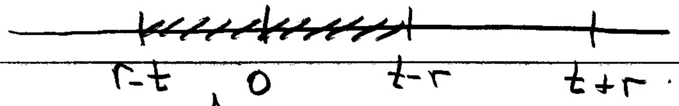
since the initial data are

$$rM_u\Big|_{t=0} = rM_f(x, r)$$

$$\partial_t (r M_u)\Big|_{t=0} = r M_g(x,r).$$

Since M_u is <u>even</u> in r, we can rewrite the soln formula as

$$r M_u(x,r;t) = \frac{1}{2}\Big[(t+r) M_f(x,r+t) - (t-r) M_f(x,t-r)\Big] + \frac{1}{2}\int_{t-r}^{t+r} \xi M_g(x,\xi)\, d\xi$$



[integral of $\xi M_g(x,\xi)$ here is 0]

Taking the limit as $r \to 0$ we get

$$u(x,t) = \partial_t\big[t M_f(x,t)\big] + t M_g(x,t)$$

which was our 2nd form of the 3D soln formula. <u>Done</u>.