PDE - Lecture 1, 9/2/2014

See syllabus for prerequisites, semester plan, etc.

While we will discuss many explicit soln formulas, our interest is not so much in being able to "write down" solns. Rather, it is in understanding properties of solns, and methods for finding or analyzing solns. Also in understanding various classes of pde's – where they're from, how solns behave, and how they differ from one another.

Many books start with 1st order eqns + method of characteristics. We'll do those topics later. Instead, I'll start with diffusion eqns like $u_t = \Delta u$.

Motivation #1 (see eg Guenther + Lee §1.4): problems involving convection + diffusion.

$\rho(x,t)$ = concentration of substance that's being carried by a flow, but also diffusing (eg ink in water)

$\vec{v}(x,t)$ = velocity of the flow (assumed to be known)

Conservation of mass says: for any fixed region D,

$$\frac{d}{dt}\int_D \rho\, dx = -\int_{\partial D} \vec{q}\cdot\vec{n}$$

where $\vec{q}$ is mass flux (so $\vec{q}\cdot\vec{n}$ is mass per unit area per unit time across surface perp to $\vec{n}$). Rewrite as

$$\int_D (\rho_t + \operatorname{div}\vec{q})\, dx = 0 \quad \text{for any } D$$

to see

$$\rho_t + \operatorname{div}\vec{q} = 0$$

In absence of diff'n, $\vec{q} = \rho\vec{v}$, and we get the "transport eq'n" $\rho_t + \operatorname{div}(\rho\vec{v}) = 0$.

Simplest (linear) model of diff'n (combined with convection) is

$$\vec{q} = -D\nabla\rho + \rho\vec{v}$$

(The linear law "diffusive flux $= -D\nabla\rho$" is "Fick's law"); then we get the convection-diffusion eq'n

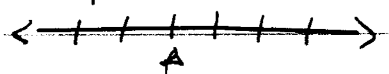
$$\rho_t - \operatorname{div}(D\nabla\rho) + \operatorname{div}(\rho\vec{v}) = 0.$$

Note that if $\vec{v} = 0$ (no flow) + $D =$ const this reduces to

$$\rho_t = D \Delta \rho .$$

<u>Motivation #2</u>: probability (cf Guenther + Lee 25.6, but my discn is somewhat different).

It is easiest to focus on spatially-discrete <u>random walks</u> (which lead to finite-difference approxns of pde's). Focus for simplicity on 1D problems.

<u>Most basic example</u>: random walker on 1D lattice of size Δx  $j\Delta x$, who flips unbiased coin at time $n\Delta t$ + goes left or right with equal prob. Let

$u_j^n = u(j\Delta x, n\Delta t)$ = prob of being at node $j\Delta x$ at time $n\Delta t$.

Then

$$u_j^{n+1} = \tfrac{1}{2} u_{j-1}^n + \tfrac{1}{2} u_{j+1}^n .$$

so

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} = \frac{(\Delta x)^2}{2\Delta t}\left[\frac{u_{j+1}^n + u_{j-1}^n - 2u_j^n}{(\Delta x)^2}\right]$$

which if $\frac{(\Delta x)^2}{2\Delta t}$ is a finite-difference discretization of $u_t = u_{xx}$. Initial cond for this evolution = initial probability density.

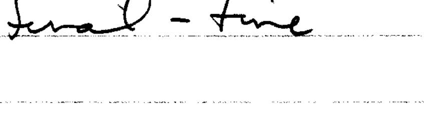
<u>Related examples:</u>

a) For the same random walker, let $z(n\Delta t)$ be his position at time $n\Delta t$, and consider the "expected final-time reward" at time $T = N\Delta t$:

$$v_j^n = v(j\Delta x, n\Delta t) = E[\varphi(z(N\Delta t))]$$ given that the walker is at location $j\Delta x$ at time $n\Delta t$

(here $n < N$, and φ is a given "final-time reward" depending only on the final-time position). It solves

$$v_j^n = \frac{1}{2} v_{j+1}^{n+1} + \frac{1}{2} v_{j-1}^{n+1}$$



ie

$$v(j\Delta x, n\Delta t) = \tfrac{1}{2} v((j+1)\Delta x, (n+1)\Delta t) + \tfrac{1}{2} v((j-1)\Delta x, (n+1)\Delta t)$$

since each time step is independent, and walker starting at $j\Delta x$ at $n\Delta t$ will be at either $(j+1)\Delta x$ or $(j-1)\Delta x$ at next time (with equal probs). Manipulation as in "most basic example" gives

$$\frac{V_j^n - V_j^{n+1}}{\Delta t} = \frac{(\Delta x)^2}{2\Delta t}\left[\frac{V_{j-1}^{n+1} + V_{j+1}^{n+1} - 2V_j^{n+1}}{(\Delta x)^2}\right].$$

which if $\frac{(\Delta x)^2}{2\Delta t} = 1$ is a finite-difference version of "heat eqn backward in time"

$$V_t + V_{xx} = 0 \quad \text{for} \quad t < T$$

The evolution of V has a final-time condn

$$V(j\Delta x, N\Delta t) = \varphi(j\Delta x)$$

(or, in continuum limit, $V(x,T) = \varphi(x)$).

Our "most basic example" is the Forward Kolmogorov eqn of our random walk (describing evolution of its probability density). Example (a) is the associated "Backward Kolmogorov eqn" (describing an expected reward at time T). The fact that the backward + forward eqns are related by $t \to -t$ is due to the simplicity of this example; in general the spatial operators are adjoints of one another.

(b) What if our random walker is restricted to the unit interval $0<x<1$, in sense that he "dies" if he reaches $x=0$ or $x=1$? Now spatial positions are x_j, $1 \le j \le N-1$, with $x_j = j/N$ (so $\Delta x = 1/N$). Analogue of our "most basic example" is

$u_j^n = u(j\Delta x, n\Delta t)$ = prob of being still alive at time $n\Delta t$ + being located at node $j\Delta x$

Previous args still apply, giving (if $\frac{(\Delta x)^2}{2\Delta t} = 1$) finite-difference approxns to $u_t = u_{xx}$, but now we have the bdry condns

$$u(0,t) = 0$$
$$u(0,t) = 0$$

as well as the initial cond
$u(x,0)$ = initial prob density.

Motivation 3: Reaction-diffusion eqns,

$$u_t = D\Delta u + f(u),$$

for example:

- simple model of population growth is

$$u_t = D\Delta u + c_0 u(1 - c_1 u)$$

("logistic growth model with diffusion") where

u = population density
D = diffusion const
c_0 = growth rate (ignoring diffusion) when density is low (so competition can be ignored)
c_1 captures fact that competition limits growth

- simple model of chemical reaction

$$u_t = \Delta u + u^2 \quad \text{or} \quad u_t = \Delta u + e^u$$

(u = nondimensionalized concentration, if growth rate is nonlinear + doesn't saturate).

Motivation 4: heat transfer (see Guenther + Lee §1.2). Rather parallel to our first motivation.

$e(x,t)$ = thermal energy per unit vol.
$q(x,t)$ = heat flux

$$\frac{d}{dt}\int_D e\,dx = -\int_{\partial D} q\cdot n\,ds + \int_D f\,dx.$$

where f = energy supplied by internal sources. Apply divergence thm as before $\Rightarrow$

$$e_t = -\operatorname{div} q + f$$

If we assume

$$e = c\,u \qquad u(x,t) = \text{temperature}$$

$$q = -k\nabla u \qquad \text{"Fick's law"}$$

Then

$$c\,u_t = \operatorname{div}(k\nabla u) + f.$$

Typical bc are

"Dirichlet" (fix u at bdry of object).

or "Neumann" (fix $q \cdot n$ at bdry of object)

<u>Notes</u>:

- Discn of reaction-diffn eqns can be done this way
- Such argts also lead easily to <u>nonlinear</u> eqns, if we replace Fick's law by something more nonlinear. For example,

$$u_t = \Delta(u^p)$$ "porous medium eqn"

comes from hypothesis $q = -\nabla(u^p) = -p u^{p-1} \nabla u$

(ie: "diffusion constant" proportional to u^{p-1}).

These motivations show that <u>both</u> bounded domains <u>and</u> whole-space problems are of interest. We'll focus on bounded domains first, since they are in some ways easier.

Uniqueness of solutions, by "energy method".

Consider $$u_t - \Delta u = f \quad \text{in } \Omega \quad (\Omega \subset \mathbb{R}^n \text{ bounded})$$
$$u = u_0 \quad \text{at } \partial\Omega$$
$$u = \varphi \quad \text{at } t=0.$$

Since eqn is linear, difference of two solns satisfies

$$\tilde{u}_t - \Delta \tilde{u} = 0 \quad \text{in } \Omega$$
$$\tilde{u} = 0 \quad \text{at } \partial\Omega$$
$$\tilde{u} = 0 \quad \text{at } t=0.$$

Multiply by $\tilde{u}$ + integ by parts to get

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |\tilde{u}|^2 + \int_\Omega |\nabla \tilde{u}|^2 = 0$$

Evidently, $\frac{d}{dt}\int |\tilde{u}|^2 \le 0$. But $\tilde{u}=0$ at $t=0$. So $\tilde{u} \equiv 0$.

Can we do something similar with a transport term? Certainly! For eqn

$$\tilde{u}_t - \Delta \tilde{u} + \operatorname{div}\left(b(x)\, \tilde{u}\right) = 0 \quad \text{in } \Omega$$

with zero initial + bdry data, mult by $\tilde{u}$ + intégn by parts gives

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |\tilde{u}|^2 + \int_\Omega |\nabla\tilde{u}|^2 \le \|b\|_\infty \int_\Omega |\tilde{u}||\nabla\tilde{u}| .$$

Now use $2|\tilde{u}||\nabla\tilde{u}| \le \alpha|\tilde{u}|^2 + \frac{1}{\alpha}|\nabla\tilde{u}|^2$ with $\frac{\|b\|_\infty}{2\alpha} = \frac{1}{2}$ (ie $\alpha = \|b\|_\infty$) to get

$$\|b\|_\infty |\tilde{u}||\nabla\tilde{u}| \le \frac{\|b\|_\infty^2}{2}|\tilde{u}|^2 + \frac{1}{2}|\nabla\tilde{u}|^2$$

so

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |\tilde{u}|^2 \le \frac{1}{2}\|b\|_\infty^2 \int_\Omega |\tilde{u}|^2$$

(after dropping a pos term from LHS). Now $\tilde{u} \equiv 0$ at $t=0$ $\Rightarrow$ $\tilde{u} \equiv 0$.

<u>Notes:</u>

1) same argt works also for $u_t - \Delta u + \bar{b}(x)\cdot\nabla u = 0$

2) preceding arguments work even in $\mathbb{R}^n$ provided the integrals converge; but that's a condition on behavior of u as $|x| \to \infty$ that we might not always want to assume.

Similar "energy-type" arg'ts can be used to assess rate at which u approaches a steady-state soln. For example:

- if $u_t - \Delta u = f$ in Ω (bounded!)
$u = u_0$ at $\partial\Omega$.

then as $t \to \infty$, u approaches the "steady state soln" $-\Delta \bar{u} = f$ in Ω.
$\bar{u} = u_0$ at $\partial\Omega$.

Accepting the existence of this steady-state soln, we can prove this by considering $\tilde{u} = u - \bar{u}$. It solves

$$\tilde{u}_t - \Delta \tilde{u} = 0 \quad \text{in } \Omega.$$
$$\tilde{u} = 0 \quad \text{at } \partial\Omega$$

(with nonzero initial cond'n $u(x,0) - \bar{u}(x)$.)
Arguing as before,

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |\tilde{u}|^2 = -\int_\Omega |\nabla \tilde{u}|^2 \, dx.$$

Now use Poincaré's ineq, which says that

(*) if $g=0$ at $\partial\Omega$ then $\int_\Omega g^2 \le C_\Omega \int_\Omega |\nabla g|^2$

[in fact, the optimal value of C_Ω is

$$\frac{1}{C_\Omega} = \min_{\substack{g=0 \\ \text{at } \partial\Omega}} \frac{\int_\Omega |\nabla g|^2}{\int_\Omega g^2} = \begin{array}{l}\text{1st eigenvalue} \\ \text{of Laplacian} \\ \text{with Dir bc,}\end{array}$$

we'll return to this later, but it's elementary for $\Omega = [0,1]$ using Fourier sine series as basis, see Guenther + Lee chaps 3+4.]

Using (*), we get

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |\tilde{u}|^2 \le -\frac{1}{C_\Omega}\int_\Omega |\tilde{u}|^2$$

so $\tilde{u} = u - \bar{u}$ has L^2 norm decaying to 0 exponentially fast as $t \to \infty$.

(Note: preceding discn assumes the source term f + bc u_0 are indep of t.)

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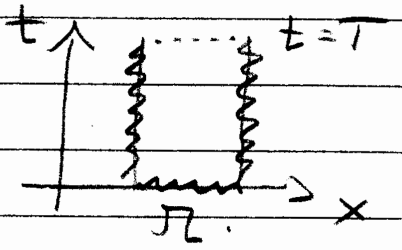
Arguments similar to the above can be

used also when the bc is of Neumann type ($\partial u/\partial n$ given at $\partial\Omega$) but there are some differences. This will be explored in HW1.

Uniqueness of solns, by "max principle".

Energy method is great, but max prin provides another very simple approach to uniqueness — and has lots of other uses as well.

Max principle (weak form, for linear heat eqn): if $u_t - \Delta u = 0$ in $\Omega \times (0,T)$, where $\Omega \subset \mathbb{R}^n$ is a bounded domain, then $\max(u)$ and $\min(u)$ are achieved either at initial time or at the spatial bdry



Pf: As a first pass let's show that if $u_t - \Delta u < 0$ (strict $\neq$) then $\max u$ is achieved at initial time or spatial bdry. In fact (assuming u is C^2)

interior max $\Rightarrow$ $u_t = 0$, $\nabla u = 0$, $\Delta u \le 0$
final time max $\Rightarrow$ $u_t \ge 0$, $\nabla u = 0$, $\Delta u \le 0$

both of which contradict $u_t - \Delta u < 0$.

Now the <u>second pass</u>: if we know only $u_t - \Delta u \le 0$ then for $\varepsilon > 0$ consider

$$u_\varepsilon(x,t) = u(x,t) - \varepsilon t.$$

It has $\partial_t u_\varepsilon - \Delta u_\varepsilon < 0$, so by our "1st pass" result

$$\max_{\substack{x \in \Omega \\ 0 \le t \le T}} (u - \varepsilon t) \le \max_{\substack{\text{initial} \\ +\text{spatial bdry}}} (u - \varepsilon t)$$

Now let $\varepsilon \to 0$.

For assertion abt $\min(u)$, apply what we just did to $-u$, or else argue as above with $\varepsilon < 0$ and $u_t - \Delta u \ge 0$.

Max prin implies uniqueness by applying it to difference of two solns.

Essentially same args work for much more general eqn

$$u_t - \sum a_{ij} \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum b_i \frac{\partial u}{\partial x_i} = 0$$

provided $a_{ij}(x,t)$ is <u>nonneg</u> symmetric-matrix-valued fn, i.e.

$$\sum a_{ij}(x,t)\, \xi_i \xi_j \geq 0 \quad \text{all } x \in \Omega$$
$$+ \text{ all } \xi \in \mathbb{R}^n$$

For uniqueness with a Neumann bc ($\partial u/\partial\nu = 0$) there is also a max-min-type arg, though pf is slightly different due to the different bc. See HW 1.

Max min is a <u>great</u> tool for proving qualitative results abt solns of heat eqns. For an excellent (well-organized + concise) discn see §5.3.3-5.3.5 of Qing Han's book (pp 185-197).

For a (more basic) treatment, close to mine, see Guenther + Lee §5.2