PDE-I Midterm

October 22, 2013

- This is a closed-book exam, however you may use one page of notes prepared in advance (8.5×11, both sides, any font).
- You'll be given plain white paper for scratch, and a bluebook for your solutions. I will grade *only* the bluebook, not the scratch paper.
- There are 8 questions, worth 15 points each. Some may be harder or more time-consuming than others. Do the ones you find easiest first.
- Explain all answers (at least briefly). Partial credit will be given for correct ideas.

(1) Consider the PDE

$$u_t - \Delta u + u - 1 = 0$$

in a bounded domain Ω, with the homogeneous Dirichlet boundary condition $u = 0$ at $\partial\Omega$. For what functional $E[u]$ is this the L^2 steepest descent?

(2) Let Ω be a bounded domain in R^n, and let $f : \Omega \to R$ be a function defined on Ω. Consider the solution of

$$u_t - \Delta u = f \quad \text{for } t > 0 \text{ and } x \in \Omega$$

with the homogeneous Dirichlet boundary condition $u = 0$ at $\partial\Omega$, and initial condition $u = u_0$ at $t = 0$. Show that

$$\int_\Omega |u(x,t) - u_*(x)|^2 \, dx \leq Ce^{-\lambda t}$$

where C and λ are suitable constants and u_* solves $-\Delta u_* = f$ in Ω with $u_* = 0$ at $\partial\Omega$. (You should assume that such a function u_* exists.)

(3) Let Ω be a bounded domain in R^n. Consider the solution of

$$u_t - \Delta u = 5u$$

in Ω, with the homogeneous Neumann boundary condition $\partial u/\partial n = 0$ at $\partial\Omega$. Characterize the initial data $u_0 = u(x, 0)$ for which the solution $u(x,t)$ stays bounded as $t \to \infty$.

(4) Suppose u is a bounded solution of the heat equation

$$u_t - \Delta u = 0$$

in all R^n, with L^1 initial data $u_0(x) = u(x, 0)$. State and prove an optimal decay estimate of the form

$$\sup_x |u(x,t)| \leq Ct^{-\alpha} \int |u_0(x)| \, dx.$$

(5) Suppose u is C^2 and harmonic in a neighborhood of $x_0 \in R^n$. Show that u is in fact C^∞ near x_0. (Hint: one method is to consider the Laplacian of $u\phi$, where ϕ is supported in the region where u is harmonic and $\phi \equiv 1$ near x_0.)

(6) Use the Mean Value Principle to show that if u is harmonic in the ball B_r then

$$|\nabla u(0)| \leq \frac{C}{r} \max_{x \in \partial B_r} |u(x)|$$

(7) Let Ω be a bounded domain in R^n, and consider solutions of $\Delta u = f$ in Ω with $u = 0$ at $\partial\Omega$. Using a maximum-principle-based argument, show that

$$\max_{\Omega} |u| \leq C \max_{\Omega} |f|$$

where C is a constant depending only on Ω.

(8) Let B be the unit ball in R^2, and consider the boundary value problem

$$\Delta u = f \text{ in } B, \text{ with } u = g \text{ at } \partial B.$$

Show that if f and g are even functions of x_2 (in the sense that $f(x_1, x_2) = f(x_1, -x_2)$ for $x \in B$ and $g(x_1, x_2) = g(x_1, -x_2)$ for $x \in \partial B$), then u is also an even function of x_2.