PDE I – Problem Set 9. Distributed 11/15/2013, due 11/26/2013.

(1) Solve the 3D wave equation in $\{r \neq 0,\ t > 0\}$ with zero initial conditions and the "boundary condition"

$$\lim_{r \to 0} 4\pi r^2 u_r(r,t) = g(t),$$

where g is a specified function of t. (Assume that $g(0) = g'(0) = g''(0) = 0$, so that the "boundary data" are compatible with the initial conditions).

(2) Recall that the solution of the 2D wave equation with initial condition $u = 0$, $u_t = g$ at $t = 0$ is

$$u(x_1, x_2, t) = \int_{|y-x|<t} \frac{g(y)}{\sqrt{t^2 - |y-x|^2}}\, dy_1\, dy_2.$$

Let's consider the behavior of the solution for large t.

(a) Show that if g has compact support, then for any fixed $x \in R^2$ we have $|u(x,t)| \leq C/t$ as $t \to \infty$ with x held fixed.

(b) Now consider the special case

$$g(x) = \begin{cases} 1 & \text{for } |x| < 1 \\ 0 & \text{for } |x| > 1. \end{cases}$$

This g is not C^2, but we can still consider the function u defined by the solution formula. Show that if e is any unit vector, $u(te, t)$ is of order $t^{-1/2}$ as $t \to \infty$.

(c) Show that if g is smooth and compactly supported, then $\max_{x \in R^2} |u(x,t)| \leq C/\sqrt{t}$.

[Note how different 2D is from 3D. In 3D with compactly supported initial data, the analogue of (a) is that $u(x,t)$ vanishes for sufficiently large t when x is held fixed, and the analogue of (c) is that $\max_{x \in R^3} |u(x,t)| \leq C/t$.]

(3) Use the "method of descent" to derive the solution formula for the 1D wave equation from the solution formula for the 2D wave equation.

(4) Pursue this alternative method for finding a solution formula for the wave equation, using the Fourier transform in R^n

$$\hat{w}(\xi) = (2\pi)^{-n/2} \int e^{-ix\cdot\xi} w(x)\, dx$$

and the fact that a function can be recovered from its Fourier transform,

$$w(x) = (2\pi)^{-n/2} \int e^{ix\cdot\xi} \hat{w}(\xi)\, d\xi.$$

(a) Show that if $u_{tt} = \Delta u$ then the Fourier transform (in space only) $\hat{u}(t,\xi)$ satisfies

$$\begin{aligned} \hat{u}_{tt} &= -|\xi|^2 \hat{u} && \text{for } t > 0 \\ \hat{u}(0,\xi) &= \hat{f}(\xi) && \text{at } t = 0 \\ \hat{u}_t(0,\xi) &= \hat{g}(\xi) && \text{at } t = 0 \end{aligned}$$

where f and g are the initial data of u.

(b) Conclude that

$$\hat{u}(t,\xi) = \cos(|\xi|t)\hat{f}(\xi) + \frac{\sin(|\xi|t)}{|\xi|}\hat{g}(\xi).$$

(c) Check that in one space dimension this yields the familiar solution formula

$$u(x,t) = \frac{1}{2}[f(x+t) + f(x-t)] + \frac{1}{2}\int_{x-t}^{x+t} g(\xi)\, d\xi.$$

[In doing this problem, you should assume that all the Fourier transforms and inverse Fourier transforms exist, that if you need to differentiate under an integral you may do so, etc.]

(5) [Before doing this problem, you should read how the method of spherical means leads to the solution formula for the 3D wave equation. This can be found for example in my Lecture 9 notes, and in Section 5.1 of Fritz John's book.] Use the method of spherical means to solve the wave equation in R^5, by proceeding as follows.

(a) Consider the function $N(x;r,t)$ defined by

$$N(x;r,t) = r^2\partial_r M_u + 3rM_u$$

where M_u is the spherical mean of u. Show that N solves the 1D wave equation in r and t.

(b) Show that

$$\begin{aligned} u(x,t) &= \lim_{r\to 0}\frac{1}{3r}N(x;r,t) \\ &= \left(\frac{1}{3}t^2\partial_t + t\right)M_g(x,t) + \partial_t\left[\left(\frac{1}{3}t^2\partial_t + t\right)M_f(x,t)\right] \end{aligned}$$

where f and g are the initial values of u and u_t.

(c) Verify that (as in 3D) the true domain of dependence is a sphere, not a ball.