PDE I – Problem Set 8. Distributed 11/07/2013, due 11/19/2013.
Problem 4 corrected 11/15 – the original version had "incoming" and "outgoing" reversed.
Problem 7 corrected 11/18 – the original version lacked a factor of $1/2$ *in the definition of* $e(t)$*.*

(1) Consider a solution of the 1D wave equation

$$u_{tt} - u_{xx} = 0$$

with compactly supported initial data

$$u = f \text{ and } u_t = g \text{ at } t = 0.$$

We know that the sum of the "kinetic energy" $k(t) = \int u_t^2\,dx$ and the "potential energy" $p(t) = \int u_x^2\,dx$ is independent of t. (Note that both are finite, since u is compactly supported in space at every time.) Show that when t is sufficiently large these two "energies" are equal, in other words $k(t) = p(t)$.

(2) Let α be constant, $\alpha \neq -1$. Consider the wave equation on $x > 0$, $t > 0$ with the following data:

$$\begin{aligned} u_{tt} - u_{xx} &= 0 && \text{for } x > 0, t > 0 \\ u_t &= \alpha u_x && \text{at } x = 0 \\ u &= f(x) && \text{at } t = 0 \\ u_t &= g(x) && \text{at } t = 0. \end{aligned}$$

Assume that f and g vanish near $x = 0$. Give a formula for u. (Hint: start with $u = F(x+t) + G(x-t)$; find F and G.) Why must $\alpha = -1$ be excluded? Can you do something even if f and g don't vanish near $x = 0$?

(3) Consider solutions of $u_{tt} = \Delta u$ in $R^3 \times R_+$ which are radially symmetric in x. Show that the general solution is

$$u = \frac{F(r+t) + G(r-t)}{r}$$

where $r = |x|$. Show that for initial data of the special form

$$u = 0, \quad u_t = g(r) \quad \text{at } t = 0$$

(with g an even function of r) the solution is

$$u = \frac{1}{2r}\int_{r-t}^{r+t} \rho g(\rho)\,d\rho. \tag{1}$$

Consider the case when

$$g(r) = \begin{cases} 1 & 0 < r < a \\ 0 & r > a. \end{cases}$$

Show there is a discontinuity at the origin at time $t = a$. (Assume u is represented by (1), though it is not C^2 so the derivation of this formula is not strictly speaking applicable.)

(4) Let's look further at the radial solutions of the wave equation discussed in Problem 3.

(a) Let $G(r) = 1$ for $r \leq 1$ and 0 for $r > 1$. Where is $u(r,t) = G(r-t)/r$ nonzero? (This solution can be viewed as an "outgoing wave.")

(b) Discuss the character of $G(r-t)/r$ for a general function G.

(c) Let $F(r) = 1$ for $r > 100$ and $F = 0$ for $r < 100$. Where is $u(r,t) = F(r+t)/r$ nonzero? (This solution can be viewed as an "incoming wave.")

(d) Discuss the character of $F(r+t)/r$ for a general function F.

(5) The pde $u_{tt} - u_{xx} + m^2 u = 0$ with $m \neq 0$ is known as the (one-dimensional) Klein-Gordon equation.

(a) For what choice of potential energy do we have that the kinetic + potential energy is constant in time?

(b) Show that, as for the wave equation, if at time 0 we have $u = u_t = 0$ in the interval $(-a, a)$, then u vanishes in the triangle $\{|x| \leq a - t\}$.

(6) Let $u_{tt} = \Delta u$ in $R^n \times R_+$. Show that

$$\sigma = \left[-2u_t \nabla u, (u_t^2 + |\nabla u|^2)\right]$$

is divergence-free (as a vector field in R^{n+1}). Use this to show that $u(x,t)$ depends only on the initial data in the ball $\{y : |y - x| \leq t\}$.

(7) Let Ω be a bounded domain in R^n, and suppose u solves

$$u_{tt} - \Delta u = f(x) \text{ for } x \in \Omega,\ t > 0.$$

(a) Suppose the boundary condition is $u = 0$ at $\partial\Omega$. Estimate $e(t) = \int_\Omega \frac{1}{2}u_t^2 + \frac{1}{2}|\nabla u|^2\, dx$ in terms of its value at time 0, and the L^2 norm of f. Your answer should have the character of a well-posedness result; in other words, it should show that if $e(0)$ and $\|f\|_{L^2}$ are small enough then $e(t)$ is small. (Hint: start by considering $\frac{d}{dt}e(t)$.)

(b) Same question, when the boundary condition is $du/dn = 0$ at $\partial\Omega$.

(c) Can you do something similar when the boundary condition is $du/dn + au = 0$ with $a > 0$? (Hint: You'll want to change the definition of $e(t)$ in this case. The good choice of $e(t)$ should be independent of time when $f = 0$.)