PDE I – Problem Set 7. Distributed 10/30/2013, due 11/12/2013.

(1) If u is harmonic on $B_r(0) \subset R^n$ with $u = g$ at $|x| = r$, it can be represented using Poisson's formula:

$$u(x) = \frac{r^2 - |x|^2}{n\alpha(n)r} \int_{\partial B_r(0)} \frac{g(y)}{|x-y|^n}.$$

(As we have discussed in class, this follows from the explicit Green's function for a ball; for the purposes of this problem you should take it as known.) Use this to show that if u is harmonic and nonnegative on $B_r(0)$ then

$$r^{n-2}\frac{r - |x|}{(r+|x|)^{n-1}}u(0) \le u(x) \le r^{n-2}\frac{r+|x|}{(r-|x|)^{n-1}}u(0).$$

(This is an explicit version of Harnack's inequality.)

(2) Recall that a periodic function on R^n (with period 1 in each variable) has a Fourier series:

$$u(x) = \sum_{k \in Z^n} \hat{u}(k) e^{2\pi i k\cdot x}.$$

Let's use this to study the inhomogeneous Laplace equation

$$\Delta u = f$$

with periodic boundary conditions (we assume f is periodic, and we seek a solution with u periodic):

(a) What consistency condition should f satisfy? Show by an energy argument that u is unique up to an additive constant.

(b) Express the Fourier series of u in terms of that of f.

(c) Show that for each i, j,

$$\int_Q \left| \frac{\partial^2 u}{\partial x_i \partial x_j} \right|^2 \le C \int_Q |f|^2$$

where $Q = [0,1]^n$ is the period cell. Can you identify the optimal value of C?

(3) The Lecture 7 notes give a variational principle for solving the Neumann boundary value problem $\Delta u = f$ in Ω with $\partial u/\partial n = g$ at $\partial\Omega$ (provided of course that f and g are consistent). This problem shows that one cannot solve that PDE problem by instead imposing the boundary condition as a constraint. For simplicity let's work in 1D, taking $\Omega = (0,1)$; and let's take $f = 0$. *Here's the question*: show that for an $a, b \in R$, the (misguided) variational problem

$$\min_{u_x(0)=a,\, u_x(1)=b} \int_0^1 u_x^2$$

has minimum value 0. [Food for thought: why is it OK to fix $u|_{\partial\Omega}$, as we do for a Dirichlet boundary condition, though this problem shows that it is not OK to fix $\partial u/\partial n|_{\partial\Omega}$?]

(4) Use the convexity of

$$E[u] = \int_\Omega \frac{1}{2}|\nabla u|^2 + \frac{1}{4}u^4\,dx$$

to prove that there can be at most one solution of $-\Delta u + u^3 = 0$ in Ω with a given Dirichlet boundary condition $u = g$ at $\partial\Omega$.

(5) Let Ω be a bounded domain in R^n, and consider the operator

$$Lu = \sum_{i,j=1}^{n} a_{ij}(x)\frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i(x)\frac{\partial u}{\partial x_i}$$

where $a_{ij}(x)$ and $b_i(x)$ are continuous and $a_{ij} = a_{ji}$. Assume moreover that there is a positive lower bound on the eigenvalues of a_{ij}, i.e. that $\sum_{i,j} a_{i,j}(x)\xi_i\xi_j \geq c_0|\xi|^2$ for all $x \in \Omega$ and all $\xi \in R^n$, for some $c_0 > 0$. Show that

(a) if u is C^2 and $Lu \geq 0$ in Ω then $\max_{x\in\Omega} u(x) = \max_{x\in\partial\Omega} u(x)$;

(b) if If u is C^2 and $Lu \leq 0$ in Ω then $\min_{x\in\Omega} u(x) = \min_{x\in\partial\Omega} u(x)$.

(Hint: consider, for sufficiently large λ, the function $u_\epsilon = u(x) \pm \epsilon e^{\lambda x_1}$.)

(6) Use problem 5 to show that if Ω is a bounded domain in R^n and $F : R^n \to R$ is smooth then there can be at most one solution of $\Delta u = F(\nabla u)$ with a given Dirichlet boundary condition $u = g$ at $\partial\Omega$. (Hint: By Taylor's theorem with remainder, $F(\xi) - F(\eta) = \left(\int_0^1 \nabla F(\eta + t(\xi - \eta))\,dt\right) \cdot (\xi - \eta)$.)

(7) A question about the finite element method:

(a) Explain why if u and v are piecewise linear on $[0,1]$, determined by their nodal values u_j, v_j at $x_j = j/N$, then integration gives

$$\int_0^1 uv\,dx = \frac{1}{N}\langle K\vec{u}, \vec{v}\rangle$$

where K is a symmetric matrix, $\vec{u} = (u_0, u_1, \ldots, u_N)$ and $\vec{v} = (v_0, v_1, \ldots, v_N)$. What is K?

(b) With the same notation as in (a), express $\int_0^1 u_x^2\,dx$ in terms of the nodal values of u.