PDE I – Problem Set 6. Distributed Thurs 10/10/2013, due Fri 10/18/2013 by 5pm (in my WWH lobby mailbox or under my office door). *No extensions.*

(1) In proving the "weak form" of the maximum principle, we actually showed a little more, namely that

- if $\Delta u \geq 0$ in Ω then u achieves its max at $\partial\Omega$, and
- if $\Delta u \leq 0$ in Ω then u achieves its min at $\partial\Omega$,

where Ω is a bounded domain in R^n. Using these, find an explicit constant C such that

$$\max_B |u| \leq C \max_B |f|$$

when $B = B_1(0)$ is the unit ball in R^n, and u solves the boundary value problem

$$\Delta u = f \text{ in } B \text{ with } u = 0 \text{ at } \partial B. \tag{1}$$

(2) Problem 1 concerns how the maximum principle can be used to prove well-posedness of the boundary value problem described by eqn (1). This problem concerns how the energy method gives an alternative approach to well-posedness.

(a) Poincare's inequality says that if $u = 0$ at $\partial\Omega$ then $\int_\Omega u^2 \leq C \int_\Omega |\nabla u|^2$ (here Ω is a bounded domain in R^n, and the constant C depends on Ω). Prove it. [Hint: we can extend u by 0 outside Ω, so it's defined in a cube in R^n. Therefore it suffices to prove the inequality when Ω is a cube.]

(b) Using Poincare's inequality, show that if u solves eqn (1) then $\int_\Omega |\nabla u|^2 \leq \int_\Omega f^2$. [Hint: multiply the equation by u and integrate.]

(3) We proved the maximum principle for harmonic functions on a bounded domain. When the domain is unbounded an additional hypothesis is needed; for example, in the halfspace $x_n > 0$ the linear function $u(x) = x_n$ is harmonic but doesn't achieve its maximum on the boundary. Let's focus for simplicity on the 2D halfspace $\Omega = \{(x_1, x_2) : x_2 > 0\}$. Show that if u is C^2 and harmonic on this Ω and continuous up to the boundary, and if in addition u is uniformly bounded from above, then $\max_\Omega u = \max_{\partial\Omega} u$. [Hint: for $\epsilon > 0$, consider the harmonic function $u(x) - \epsilon \log(x_1^2 + (x_2+1)^2)^{1/2}$. Apply the maximum principle to the region where $x_1^2 + (x_2+1)^2 < a^2$ and $x_2 > 0$, with a sufficiently large. Then let $\epsilon \to 0$.]

(4) Consider the quadrant $\{x > 0, y > 0\}$ in the $x - y$ plane. What is its Green's function?

(5) Let u solve $\Delta u = 0$ in the n-dimensional halfspace $\{x_n > 0\}$, with Dirichlet data $u = g$ at $\{x_n = 0\}$. Suppose g is bounded and $g(z) = |z|$ when z is near 0. Show that ∇u is unbounded near $x = 0$. (Hint: estimate $[u(\lambda e_n) - u(0)]/\lambda$, using Poisson's formula for a halfspace; here e_n is the unit vector in the x_n direction.)