**PDE I – Problem Set 5.** Distributed Wed 10/2/2013, due Tues 10/15/2013. *No extensions beyond Friday 10/18/2013 at 5pm.* **Typos corrected in problem 5 (an $f$ was missing before) and problem 6 (a sign was wrong before).**

(1) We discussed how variational principles lead to PDE's; here is another example. Let $\Omega$ be a bounded domain in $R^n$, and consider the variational problem of minimizing the "Rayleigh quotient"

$$\min_{u=0 \text{ at } \partial\Omega} \frac{\int_\Omega |\nabla u|^2}{\int_\Omega u^2}.$$

Assume that a minimizer exists and is $C^2$. Show that it must be a Dirichlet eigenfunction of the Laplacian, i.e. a nonzero solution of

$$-\Delta u = \lambda u \quad \text{in } \Omega, \text{ with } u = 0 \text{ at } \partial\Omega.$$

Conclude that the minimum value of the Rayleigh quotient is equal to the smallest Dirichlet eigenvalue.

(2) Show that if $u$ is a $C^2$ harmonic function defined on a region in $R^n$, then

$$v(x) = |x|^{2-n} u\left(\frac{x}{|x|^2}\right)$$

is harmonic on the region where it is defined. (Hint: while this can be done by writing the Laplacian in polar coordinates, an alternative – in my view easier – argument uses the fact that $u$ is harmonic on $\Omega$ if and only if $\int_\Omega \langle \nabla u, \nabla \phi \rangle = 0$ for all $\phi$ such that $\phi = 0$ at $\partial\Omega$.)

(3) (Han, Section 4.5). Show that if $u$ is harmonic on all $R^n$ and $\int_{R^n} |u|^p \, dx < \infty$ for some $p > 1$, then $u$ is identically zero.

(4) Suppose that $u$ is harmonic on all $R^n$ and $|u(x)| \leq C|x|^k$, where $C$ is a constant and $k$ is a positive integer. Show that $u$ is a polynomial of degree at most $k$. (Hint: in the course of proving Liouville's theorem in the Lecture 5 notes, I showed that if $u$ is harmonic in $B(x_0, r)$, then $|\nabla u(x_0)| \leq C r^{-1} \max_{y \in B(x_0,r)} |u(y)|$. Use this estimate.)

(5) (From Evans, Section 2.5.) Adjust the proof of the mean value theorem to show that for $n \geq 3$, if $-\Delta u = f$ on $B(0, r)$ and $u = g$ at the boundary, then

$$u(0) = \frac{1}{|\partial B(0,r)|} \int_{\partial B(0,r)} g \, \text{darea} + \frac{1}{n(n-2)\alpha_n} \int_{B(0,r)} \left( \frac{1}{|x|^{n-2}} - \frac{1}{r^{n-2}} \right) f \, \text{dvol}.$$

(6) Show that the function $\Phi(x) = -\frac{1}{2}|x|$ is a fundamental solution of the 1D Laplacian, in the sense that for any compactly supported function $f : R \to R$, the function $u(x) = \int \Phi(x - y) f(y) \, dy$ solves $-u_{xx} = f$.

(7) (From John, Section 4.1.) Newton's law of gravitation asserts that if $\Omega$ is a three-dimensional body with density $\mu = \mu(x)$, then the force it exerts on a unit mass located at a point $y \in R^3$ is the vector

$$F(y) = \gamma \int_\Omega \frac{\mu(x)(x-y)}{|x-y|^3}\, dx,$$

where $\gamma$ is a universal constant.

(a) Show that $F = \nabla u$, where $u$ (the "gravitational potential") is given by

$$u(y) = \gamma \int_\Omega \frac{\mu(x)}{|x-y|}\, dx.$$

(b) Show that the force $F(y)$ exerted by $\Omega$ on a far away unit mass is approximately the same as if the total mass of $\Omega$ were concentrated at its center of gravity

$$x_0 = \frac{\int_\Omega \mu(x) x\, dx}{\int_\Omega \mu(x)\, dx}.$$

(Hint: approximate $|x-y|^{-3}$ by $|x_0-y|^{-3}$.)