PDE I – Problem Set 4. Distributed Tues 9/24/2013, due Tues 10/8/2013.

(1) Let Ω be a bounded domain in R^n. The Neumann Green's function $N(x, y, t)$ is the analogue of the Dirichlet Green's function, but using the boundary condition $\partial u/\partial n = 0$ at $\partial\Omega$; its defining property is that the solution of $u_t - \Delta u = 0$ in Ω with $\partial u/\partial n = 0$ at $\partial\Omega$ and $u = u_0$ at $t = 0$ is $u(x, t) = \int_\Omega N(x, y, t)u_0(y)\, dy$. (Remark: $N(x, y, t)$ is symmetric in x and y; the proof is parallel to what we did in class for the Dirichlet Green's function $G(x, y, t)$.) Show that the solution of

$$\begin{aligned} u_t - \Delta u &= 0 \quad \text{for } x \in \Omega, t > 0 \\ \partial u/\partial n &= g \quad \text{for } x \in \partial\Omega \\ u &= 0 \quad \text{at } t = 0 \end{aligned}$$

is given by

$$u(x, t) = \int_0^t \int_{\partial\Omega} N(x, y, t - s)g(y, s)\, dy\, ds.$$

(2) Show that if a smooth function u_* minimizes

$$E[u] = \int_\Omega \frac{1}{2}|\nabla u|^2 + \int_\Omega uf + \frac{1}{2}\beta \int_{\partial\Omega} |u|^2$$

then it solves $\Delta u_* = f$ in Ω, with the boundary condition $\frac{\partial u_*}{\partial n} + \beta u_* = 0$ at $\partial\Omega$. (Hint: the function $t \mapsto E[u_* + tv]$ is minimized at $t = 0$. Since we have not imposed a boundary condition on u, there is no boundary condition restricting the choice of v.)

(3) Let $f(\theta)$ be a periodic function on the unit circle in the plane, with Fourier series

$$f(\theta) = \sum_{n=0}^{\infty} a_n \cos(n\theta) + b_n \sin(n\theta).$$

Assume that $f_{\theta\theta}$ is uniformly bounded. Show that

$$u = \sum_{n=0}^{\infty} [a_n \cos(n\theta) + b_n \sin(n\theta)]r^n$$

solves Laplace's equation in the disk $r < 1$ with boundary condition f.

(4) Give a similar method for solving the Neumann problem

$$\begin{aligned} \Delta u &= 0 \quad \text{in the disk } x^2 + y^2 < 1 \\ \partial u/\partial n &= g \quad \text{at the circle } x^2 + y^2 = 1. \end{aligned}$$

(5) Now consider solving $\Delta u = 0$ with both Dirichlet and Neumann data imposed at the unit circle:

$$u = f \text{ and } \frac{\partial u}{\partial n} = g \text{ at } x^2 + y^2 = 1.$$

Let's explore whether there a solution in the *punctured* disk $0 < x^2 + y^2 < 1$.

(a) Show that there are plenty of examples where the answer is yes, by considering the Laurent expansion of a complex function that's analytic in the punctured disk.

(b) Show that there are plenty of examples where the answer is no (here, too, I suggest using some facts from complex variables).

(c) How should the Fourier series of f and g be related, if there is to be a solution that's harmonic on the whole disk $x^2 + y^2 < 1$?

(6) Recall that if f is a complex analytic function then its real and imaginary parts are harmonic. Using problems (3) and (4), discuss how conformal mapping can be used to solve Laplace's equation in a (simply-connected) plane domain with either Dirichlet or Neumann boundary data.

(7) For any α between 0 and 2π, consider Laplace's equation $\Delta u = 0$ in the pie-shaped region $\Omega = \{re^{i\theta} : 0 < r < 1, 0 < \theta < \alpha\}$, with $\frac{\partial u}{\partial n} = 0$ at the straight parts of the boundary (the segments $\theta = 0$ and $\theta = \alpha$) and $u = f(\theta)$ at the curved part (the arc $e^{i\theta}, 0 < \theta < \alpha$). Determine the character of the singularity (if any) at the origin.

(8) Let f and u be as in Problem 3. What condition on the Fourier series of f is equivalent to $\int_{x^2+y^2<1} |\nabla u|^2 \, dx \, dy < \infty$?