PDE I – Problem Set 3. Distributed Wed 9/18/2013, due Tues 10/1/2013.
Notes added 9/27/2013: (a) While this HW can be turned in 10/1, students have an automatic extension to 10/8. No extensions beyond 10/8 will be granted. (b) The hint on 2(c) is perhaps misleading. You won't have to look far to find a nonzero solution of the heat equation that's smooth for $t > 0$, decays to 0 as $|x| \to \infty$, and has $u(x,t) \to 0$ a.e. as $t \to 0$.

(1) This problem asks you to explain two assertions made at the end of the Lecture 2 notes.

(a) Consider the PDE $u_t = \Delta u + u^3$ in a bounded domain Ω, with $u = 0$ at $\partial\Omega$. Show that u evolves by "steepest descent" (with respect to the L^2 norm) for the functional $E[u] = \int_\Omega \frac{1}{2}|\nabla u|^2 - \frac{1}{4}u^4\, dx$.

(b) Consider the PDE $u_t = \text{div}\,(|\nabla u|^2 \nabla u)$ in a bounded domain Ω, with $u = 0$ at $\partial\Omega$. Show that u evolves by "steepest descent" (with respect to the L^2 norm) for the functional $F[u] = \int_\Omega \frac{1}{4}|\nabla u|^4$.

(2) Consider the heat equation $u_t = u_{xx}$ on R, with the "Heaviside function" as initial data:

$$u(x,0) = \begin{cases} 0 & \text{if } x < 0 \\ 1 & \text{if } x > 0 \end{cases}$$

(a) Show by integration against the fundamental solution that

$$u(x,t) = N(x/\sqrt{2t})$$

where N is the cumulative normal distribution

$$N(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-s^2/2}\, ds.$$

(b) Argue that this calculation is legitimate (i.e. u solves the heat equation, and it has the desired initial data) although the Heaviside function is neither continuous nor compactly supported.

(c) If $u_t - u_{xx} = 0$ for $t > 0$, and $u(x,t) \to 0$ almost every where as $t \to 0$, should we expect in general that $u = 0$? (Hint: use part (a) to give a counterexample.)

(3) Recall that for the heat equation in a bounded domain Ω with the Dirichlet boundary condition $u = 0$ at $\partial\Omega$, the solution decays exponentially to 0 as $t \to \infty$. Let's explore what happens in all space, focusing for simplicity on one space dimension:

$$\begin{aligned} u_t - u_{xx} &= 0 \quad \text{for } t > 0,\ x \in R \\ u &= u_0(x) \quad \text{at } t = 0. \end{aligned}$$

(a) Show that if u_0 is bounded and continuous, and $\int_{-\infty}^{\infty} |u_0|\, dx < \infty$, then

$$\sup_x |u(x,t)| \le C t^{-1/2}.$$

What is the optimal value of C?

(b) Show that if $u_0 = \phi_x$ with $\int_{-\infty}^{\infty} |\phi| \, dx < \infty$ then the decay is faster:

$$\sup_x |u(x,t)| \leq Ct^{-1}.$$

What is the optimal value of C in this case?

(4) Our discussion of the heat equation on the half-line $x > 0$ with a homogeneous Dirichlet ($u = 0$ at $x = 0$) or Neumann ($u_x = 0$ at $x = 0$) boundary condition used odd or even reflection. Therefore we implicitly used the following assertions:

- If $u_0 : R \to R$ is an odd function of x, then the solution of the whole-space heat equation with initial data u_0 is an odd function of x for each t.
- If $u_0 : R \to R$ is an even function of x, then the solution of the whole-space heat equation with initial data u_0 is an even function of x for each t.

(a) Give a proof of these assertions, based on our solution formula (which gives $u(x,t)$ as the convolution of u_0 with the fundamental solution).

(b) Give a different proof of these assertions, based on a uniqueness result for solutions to the initial value problem in all space. (You need to assume something here about the behavior of u_0 and u as $|x| \to \infty$. State briefly your assumptions and the uniqueness result you use. There is more than one reasonable choice: your uniqueness result should be true, but it need not be the most general result you know.)

(5) Consider the heat equation in a the first quadrant of R^2, i.e.

$$\begin{aligned} u_t - \Delta u &= 0 \quad \text{for } x \in \Omega, t > 0 \\ u &= u_0 \quad \text{at } t = 0 \end{aligned}$$

with $\Omega = \{x_1 > 0, \; x_2 > 0\}$.

(a) Let $G(x,y,t)$ be the Green's function associated with the homogeneous Dirichlet boundary condition $u = 0$ at $\partial\Omega$. (By definition, this means that the solution of the PDE with this boundary condition has the form $u(x) = \int_\Omega G(x,y,t) u_0(y) \, dy$.) Give a formula for G.

(b) Let $H(x,y,t)$ be the Green's function associated with the homogeneous Neumann boundary condition $\frac{\partial u}{\partial n} = 0$. (By definition, this means that the solution of the PDE with this boundary condition has the form $u(x) = \int_\Omega H(x,y,t) u_0(y) \, dy$.) Give a formula for H.