PDE I – Problem Set 2. Distributed Thurs 9/13/2013, due 9/24/2013.

This problem set focuses exclusively on problems in 1D. The reason is that in 1D we can use Fourier analysis, and we know the eigenvalues and eigenfunctions of the Laplacian explicitly. As you solve each problem, however, you may wish to consider whether something similar can be expected for solutions on a bounded domain in R^n.

Problems 1–5. As we discussed in class, the solution of the heat equation $u_t - u_{xx} = 0$ on the interval $(0, \pi)$ with the homogeneous Dirichlet boundary condition $u(0,t) = u(\pi,t) = 0$ and initial condition $u_0(x)$ is

$$u(x,t) = \sum_{n=1}^{\infty} a_n e^{-\lambda_n t} \phi_n(x) \tag{1}$$

with $\phi_n(x) = \sqrt{2/\pi} \sin(nx)$ and $a_n = \int_0^\pi u_0(x)\phi_n(x)\, dx$. Problems 1–5 explore some properties of this solution formula.

(1) Show that the function $u(x,t)$ defined by (1) is C^∞ in x for each $t > 0$.

(2) Assume now that $u_0(x)$ has two derivatives, with $|u_0''(x)| \leq M$ for some constant M. Assume further that u_0 satisfies the boundary condition, i.e. $u_0(0) = u_0(\pi) = 0$.

  (a) Prove an inequality of the form $|a_n| \leq C/n^2$.

  (b) Show that as t decreases to 0, the function $u(x,t)$ defined by (1) converges uniformly to $u_0(x)$.

(3) Suppose now that $u_0(x)$ has four bounded derivatives ($|u_0''''(x)| \leq M$), and $u_0''(0) = u_0''(\pi) = 0$.

  (a) Show that as t decreases to 0, $u_{xx}(x,t)$ converges uniformly to $u_0''(x)$.

  (b) If u_0 is smooth but $u_0''(0) \neq 0$ or $u_0''(\pi) \neq 0$, is it possible that the conclusion of part (a) still holds?

(4) The solution formula (1) makes sense even when u_0 doesn't vanish at the endpoints – for example when $u_0(x) \equiv 1$.

  (a) Does $u(x,t)$ satisfy the boundary conditions $u(0,t) = u(\pi,t) = 0$ for $t > 0$?

  (b) Discuss the sense in which $u(x,t)$ approaches u_0 as $t \downarrow 0$ in this case.

(5) We discussed the fact that solving the heat equation backward in time is ill-posed. But it becomes well-posed if you know that only finitely many modes appear. Suppose you are not given u_0, but instead you are told that $u(x,1) = \sum_{n=1}^{100} b_n \sin(nx)$, and you are given $b_1, \ldots, b_{100}$ up to an error of at most 10^{-3} (in each). How precisely can you determine $u_0 = u(x,0)$?

Problems 6 and 7 concern slightly different PDE's or boundary conditions.

(6) Suppose u solves

$$u_t - u_{xx} = 10u$$

on the interval $(0, \pi)$, with the homogeneous Neumann condition $u_x = 0$ at $x = 0, \pi$. Characterize the initial data $u_0 = u(x,0)$ for which $u(x,t)$ stays bounded as $t \to \infty$.

(7) Suppose u solves

$$u_t - u_{xx} = F(x)$$

on the interval $(0, \pi)$, with the homogeneous Neumann condition $u_x = 0$ and $x = 0, \pi$. Here F is a given source term, independent of t. For which choices of F does the solution have a limit as $t \to \infty$? When the limit exists, what is it?

Problems 8 and 9 concern numerical approximation.

(8) The Lecture 2 notes discussed a continuous-time, discrete-space approximation of $u_t = u_{xx}$. When u has a homogeneous Dirichlet boundary condition, the ODE for the nodal values is

$$\dot{u}_j = \frac{u_{j-1} + u_{j+1} - 2u_j}{(\Delta x)^2} \quad j = 1, \ldots, N-1$$

with the convention that the domain is $(0, N\Delta x)$ and $u_0(t) = u_N(t) = 0$. Let's discuss its convergence as $\Delta x \to 0$.

(a) Suppose the exact solution has u^{pde}_{xxxx} bounded (uniformly with respect to space and time). Consider the error $z_j(t) = u_j(t) - u^{\text{pde}}(j\Delta x, t)$. Show that if we define $\phi_j(t)$ by

$$\dot{z}_j - \frac{z_{j-1} + z_{j+1} - 2z_j}{(\Delta x)^2} = \phi_j(t), \tag{2}$$

then we have an estimate of the form $|\phi_j| \leq C(\Delta x)^2$, with the constant C depending only on an upper bound for $|u^{\text{pde}}_{xxxx}|$. (This corrects a mistake on page 10 of the Lecture 2 notes, as originally distributed.)

(b) Show that if the RHS of (2) were zero we would have a discrete version of the maximum principle. In other words: show that if $w_j(t)$ ($j = 1, \ldots, N-1$) solves the ODE system $\dot{w}_j - \frac{w_{j-1}+w_{j+1}-2w_j}{(\Delta x)^2} = 0$ with the convention $w_0(t) = w_N(t) = 0$, then $\max_{j,t} w_j(t)$ and $\min_{j,t} w_j(t)$ are achieved either at the initial time ($t = 0$) or the spatial boundary ($j = 0$ or $j = N$).

(c) Apply part (b) to $z_j \pm C(\Delta x)^2 t$ to deduce the error estimate $|z_j(t)| \leq C(\Delta x)^2 t$.

(9) The Lecture 2 notes discussed a discrete-time, discrete-space approximation of $u_t = u_{xx}$. When u has a homogeneous Dirichlet boundary condition and the domain is $(0, \pi)$, the scheme says

$$u_j(t_{n+1}) = \alpha u_{j+1}(t_n) + \alpha u_{j-1}(t_n) + (1 - 2\alpha)u_j(t_n) \tag{3}$$

with the conventions that the spatial step is $\Delta x = \pi/N$, the times are $t_n = n\Delta t$,

$$\alpha = \frac{\Delta t}{(\Delta x)^2},$$

and $u_0(t_n) = u_N(t_n) = 0$ for all n. I told you that the scheme is stable for $\alpha \leq 1/2$ and unstable for $\alpha > 1/2$. Let's understand why.

(a) Assume $0 < \alpha \leq 1/2$. Show that for any M, if initially $\max_j |u_j(0)| \leq M$, then the estimate persists: $\max_j |u_j(t_n)| \leq M$ for each $n = 1, 2, \ldots$. (Thus, the scheme is stable in the sense that a small change in its initial data produces a small change in the solution.)

(b) Suppose $\alpha > 1/2$. Consider, for any integer k, the initial data $u_j(0) = \sin(jk\Delta x)$. (Note that it vanishes at the endpoints $j = 0, N$.) Show that the associated solution is

$$u_j(t_n) = \xi^n u_j(0)$$

where $\xi = \xi(k) = 1 - 2\alpha[1 - \cos(k\Delta x)]$.

(c) The solution identified in part (b) grows exponentially in magnitude if $|\xi| > 1$. Show that if $\alpha > 1/2$, then such growth happens when $\cos(k\Delta x)$ is close enough to -1. (Thus, the scheme is unstable in the sense that a small change in its initial data can produce a huge change in the solution after multiple time steps, even at times such that $t_n = n\Delta t$ is still quite small.)