PDE I – Problem Set 11. Distributed 12/3/2013, due 12/10/2013.
There will be no HW extensions beyond Friday 12/13 at 10am.

(1) A "viscous shock profile" for Burgers' equation is a function $U(\xi)$ with $U \to u_L$ as $\xi \to -\infty$ and $U \to u_R$ as $\xi \to +\infty$, such that $u(x,t) = U[(x-\sigma t)/\epsilon]$ solves $u_t + uu_x = \epsilon u_{xx}$. We know that if U exists then σ must be equal to $(u_L + u_R)/2$ and we must have $u_L > u_R$.

(a) Show that U does indeed exist, and it's given by

$$\begin{aligned} \xi &= \int \frac{dU}{\frac{1}{2}(U-u_L)(U-u_R)} \quad + \text{constant} \\ &= \frac{2}{u_L - u_R} \log\left(\frac{u_L - U}{U - u_R}\right) \quad + \text{constant} \end{aligned}$$

(b) Show that as $\epsilon \to 0$ (and choosing the constant properly) $\lim_{\epsilon\to 0} U[(x-\sigma t)/\epsilon]$ converges to the admissible weak solution of Burgers' equation with initial condition u_L for $x < 0$ and u_R for $x > 0$.

(c) What happens to the above construction if $u_R > u_L$ (so that no shock is expected)?

(d) Can something similar be done for any conservation law $u_t + F(u)_x = 0$?

(2) Use the method of characteristics to solve these initial value problems:

(a) $u_t + u_x = u^2, \quad u(x,0) = h(x)$

(b) $u_t + xu_x + yu_y = u, \quad u(x,y,0) = h(x,y)$

(c) $xu_t - tu_x = u, \quad u(x,0) = h(x)$.

(3) Use the method of characteristics to derive these solution formulas:

(a) $xu_x - yu_y = 0 \Longrightarrow u = f(x \cdot y)$

(b) $xu_x + yu_y = \alpha u \Longrightarrow u = (x^2+y^2)^{\alpha/2} f\left(\frac{x}{\sqrt{x^2+y^2}}, \frac{y}{\sqrt{x^2+y^2}}\right)$

(4) Consider the "equation of geometrical optics" in 2D:

$$u_x^2 + u_y^2 = 1.$$

(a) Find the associated system of ODE's describing the characteristics.

(b) Show that ∇u is constant along each characteristic.

(c) Suppose $u = 0$ on a smooth, simple closed curve S in the x, y plane. Show that $u(x) = \pm$distance to S, if u is differentiable along the shortest path from x to S.

(d) Show that u cannot be smooth throughout the bounded region determined by S.

(5) Consider the first-order PDE $u_t + H(x, \nabla u) = 0$ (this "Hamilton-Jacobi equation" arises naturally in the context of optimal control). Show that solving the PDE by the method of characteristics amounts to solving the ODE system

$$\dot{x}_i = \frac{\partial H}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial H}{\partial x_i}, \quad \dot{u} = -H + \sum_i p_i \frac{\partial H}{\partial p_i}$$

with the notation $H = H(x,p)$. (If you know some physics, you'll recognize that the equations for $x(t)$ and $p(t)$ are those of Hamiltonian mechanics.)