PDE I – Problem Set 10. Distributed 11/21/2013, due 12/3/2013.

Problem 1 gives an alternative proof of the Rankine-Hugoniot condition. Problem 2 reveals that conservation laws that are equivalent to one another for smooth solutions are not equivalent for weak solutions. Problems 3-6 ask you to work out some examples. Remember that when considering a conservation law $u_t + (F(u))_x = 0$, a shock must satisfy the Rankine-Hugoniot condition $\dot{s} = [F(u)]/[u]$. In addition, shocks must satisfy the "admissibility condition" that the characteristics go *into* the shock (equivalently: $F'(u_L) > ds/dt > F'(u_R)$). The reason for this restriction is discussed in the Lecture 10 notes (we'll discuss it in class on 11/26).

(1) Suppose $u(x,t)$ and $q(x,t)$ are smooth except at a "shock" $x = s(t)$, where u and q are both discontinuous. Suppose $u_t + q_x = 0$ away from the shock. Use the fundamental theorem of calculus to show that for any $a < s(t) < b$, the relation

$$\frac{d}{dt}\left[\int_a^{s(t)} u_L(x,t)\,dx + \int_{s(t)}^b u_R(x,t)\,dx\right] = -q_R(b) + q_L(a)$$

is equivalent to

$$[u]\dot{s} = [q].$$

(Here u_L, u_R, q_L, q_R are the restrictions of u and q to the left and right of the shock; $[u] = u_R - u_L$ and $[q] = q_R - q_L$ are the jumps across the shock.)

(2) We know that

$$u(x,t) = \begin{cases} 2 & \text{for } x < 3t/2 \\ 1 & \text{for } x > 3t/2 \end{cases}$$

is an admissible weak solution of Burgers' equation (the shock speed is $(u_L + u_R)/2 = 3/2$). Show that u is not a weak solution of the conservation law $\frac{1}{2}(u^2)_t + \frac{1}{3}(u^3)_x = 0$. (Note that for smooth nonzero u, this conservation law reduces – like Burgers' equation – to $u_t + uu_x = 0$.)

(3) Consider Burgers' equation $u_t + \frac{1}{2}(u^2)_x = 0$, with initial data

$$u(x,0) = \begin{cases} 1 & \text{for } x < -1 \\ 0 & \text{for } -1 < x < 0 \\ 2 & \text{for } 0 < x < 1 \\ 0 & \text{for } x > 1. \end{cases}$$

Find the solution for all $t > 0$.

(4) Now consider Burgers' equation with initial data

$$u(x,0) = \begin{cases} 0 & \text{for } x < 0 \\ x & \text{for } 0 < x < 1/2 \\ 1-x & \text{for } 1/2 < x < 1 \\ 0 & \text{for } x > 1. \end{cases}$$

Find the time and place of shock formation, and the velocity of the shock for all later times.

(5) This time consider Burgers' equation with initial data

$$u(x,0) = \begin{cases} 0 & \text{for } x < 0 \\ 1 & \text{for } 0 < x < L \\ 0 & \text{for } x > L. \end{cases}$$

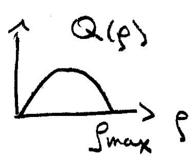
(a) Show that when t is large enough the solution has the form

$$u(x,t) = \begin{cases} 0 & \text{for } x < 0 \\ x/t & \text{for } 0 < x < s(t) \\ 0 & \text{for } x > s(t) \end{cases}$$

where $s(t)$ is an (as yet undetermined) shock locus.

(b) Use the Rankine-Hugoniot condition $ds/dt = (u_L + u_R)/2$ and the conservation law $(d/dt)\int u(x,t)\,dx = 0$ to determine the function $s(t)$.

(6) As explained in Lecture 10, a continuum model of traffic flow on a 1D road is $\rho_t + (Q(\rho))_x = 0$ where Q (the traffic flux, with units cars per unit time) should vanish at $\rho = 0$ and at $\rho = \rho_{\max}$ and be unimodal in between, as in the figure.



(a) Consider the solution with initial data

$$\rho = \begin{cases} \rho_{\max} & x < 0 \\ 0 & x > 0 \end{cases}$$

(This captures what happens after a red light turns green, if the backup behind the light is infinite.) Discuss both the space-time picture (is the solution a fan or a shock?) and the form of the function $x \mapsto \rho(x,t)$ at a given time.

(b) Let ρ_1 and ρ_2 be constants, chosen so that ρ_1 is below the value where Q is maximized, ρ_2 is above the value where Q is maximized, and $Q(\rho_1) = Q(\rho_2)$. (Note that $Q'(\rho_1) > 0$ and $Q'(\rho_2) < 0$.) Discuss the solution with initial data

$$\rho = \begin{cases} \rho_1 & x < -1 \\ \rho_2 & -1 < x < 0 \\ \rho_1 & x > 0 \end{cases}$$

(The special case $\rho_1 = 0, \rho_2 = \rho_{\max}$ captures a red light turning green, if the backup behind the light is finite.) As for (a), you should discuss both the spacetime picture (hint: there is both a fan and a shock) and the form of the function $x \mapsto \rho(x,t)$ (hint: there is a qualitative change in the form of this function at a certain time).