PDE I – Problem Set 1. Distributed Tues 9/3/2013, due 9/17/2013.

Throughout this problem set we are concerned with "classical solutions," i.e. solutions smooth enough that all convenient manipulations (integration by parts, maximum principle, etc) are justified. Except for Problem 1, this problem set is concerned exclusively with solutions of PDE's in *bounded* domains. The domain is always assumed connected. If the regularity of the boundary makes a difference, please assume it is as smooth as needed to make your argument work.

(1) Recall from Lecture 1 that for a 1D random walk with spatial step Δx, time step Δt, and probability 1/2 of going left or right, the evolution of the probability density is a finite-difference discretization of $u_t = u_{xx}$ when $\frac{(\Delta x)^2}{2\Delta t} = 1$.

   (a) Consider the biased random walk in which a walker at $j\Delta x$ moves to $(j+1)\Delta x$ with probability $\frac{1}{2} + \alpha\Delta x$ and moves to $(j-1)\Delta x$ with probability $\frac{1}{2} - \alpha\Delta x$? Assuming as before that $\frac{(\Delta x)^2}{2\Delta t} = 1$, and taking α to be constant, what PDE does the probability density solve in the continuum limit $\Delta x \to 0$?

   (b) Now suppose the bias is position-dependent; in other words, using the notation of part (a), suppose $\alpha = \alpha(j\Delta x)$ is a smooth but non-constant function of position. Extend what you found in part (a) to this case. [Warning: note that when $\alpha(x)$ is not constant, $\alpha u_x \neq (\alpha u)_x$.]

(2) We saw that for the heat equation in a bounded domain with a homogeneous Dirichlet boundary condition, $\int_\Omega |u|^2 dx$ decays exponentially to 0 as $t \to 0$. What is the analogous assertion for the heat equation in a bounded domain with a homogeneous Neumann boundary condition ($\partial u/\partial n = 0$ at $\partial\Omega$)? [Hint: There is a constant M_Ω with the following property: if $u : \Omega \to R$ has mean value 0 then $\int_\Omega u^2\, dx \leq M_\Omega \int_\Omega |\nabla u|^2\, dx$. You may use this result without proving it.]

(3) Let's explore the power of the "energy method" for proving uniqueness.

   (a) In Lecture 1's discussion of heat flow, we discussed only the cases of a Dirichlet boundary condition (fixed temperature at $\partial\Omega$) and a Neumann boundary condition (fixed heat flux at $\partial\Omega$). Another physically-natural assumption is that the heat flux from the boundary is proportional to the difference between the temperature $u(x,t)$ and some fixed constant U. Known as "Newton's law of cooling", this models loss of heat by radiation, if the far-field temperature is U. So consider the heat equation $u_t = \Delta u$ in a bounded domain $\Omega \subset R^n$, with initial condition $u = u_0(x)$ at $t = 0$ and boundary condition

$$\partial u/\partial \nu = -k(u - U) \quad \text{at } \partial\Omega,$$

   where ν is the outward unit normal and $k \geq 0$. Use the "energy method" to show that there can be at most one solution. Does a similar assertion hold also for $k < 0$?

   (b) Suppose $a(x) = a_{ij}(x)$ takes values in the class of symmetric, positive definite $n \times n$ matrices. Consider the PDE

$$u_t = \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial u}{\partial x_j}\right)$$

   in a bounded domain Ω, with initial condition $u = u_0(x)$ at $t = 0$ and a Dirichlet boundary condition $u = g$ at $\partial\Omega$. Use the "energy method" to show there can be at most one solution.

(c) Now consider the PDE

$$u_t = \sum_{i,j=1}^{n} a_{ij}(x) \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i(x) \frac{\partial u}{\partial x_i}$$

in a bounded domain $\Omega \subset R^n$, where $a(x) = a_{ij}(x)$ takes values in the class of positive definite symmetric matrices and $b(x) = b_i(x)$ takes values in R^n. Suppose the boundary condition is $u = u_0$ at $t = 0$, and the boundary condition is $u = g$ at $\partial\Omega$. Try proving uniqueness by considering the difference between two solutions, then multiplying the equation by $u(x,t)p(x)$ and integrating by parts, where p is a well-chosen function of x. Under what circumstances does this work?

(4) In Lecture 1 we gave a maximum-principle-based proof of uniqueness for the heat equation with a Dirichlet boundary condition. This problem asks you to do something similar for the heat equation with a Neumann boundary condition. Suppose $u_t - \Delta u = 0$ on $\Omega \times (0,t)$ with $\partial u/\partial n = 0$ at $\partial\Omega$. Show that u achieves its maximum and minimum values at the initial time $t = 0$. [Hint: consider $u_{\epsilon,\delta}(x) = u - \delta\phi(x) - \epsilon t$, with a suitable choices of $\phi(x), \delta$, and ϵ.]

(5) Let's look at how the maximum principle changes when the PDE has a zeroth order term. Throughout this problem, we work in a bounded domain $\Omega \subset R^n$, with Dirichlet boundary condition $u = 0$ at $\partial\Omega$ and initial condition $u(x,0) = u_0(x)$.

(a) Suppose the PDE is

$$u_t - \Delta u + c(x,t)u = 0$$

with $c(x,t) \geq 0$. Show that

$$\max u \leq \max u_0^+ \quad \text{and} \quad \min u \geq \min u_0^-$$

where u_0^+ and u_0^- are respectively the positive and negative parts of u_0.

(b) Consider the same PDE, but assume now that $c(x,t) \geq \gamma$ where γ is a positive constant. Show that $|u(x,t)| \leq Ce^{-\gamma t}$. [Hint: apply part (a) to $ue^{\gamma t}$.]

(c) Consider the same PDE, but let $c(x,t)$ be any smooth function (bounded, but possibly negative). Show that if $u_0 \geq 0$ then $u(x,t) \geq 0$ for all $x \in \Omega$ and $t > 0$. [Hint: consider $v(x,t) = e^{\lambda t}u(x,t)$ for a suitable choice of λ.]

(6) Consider two solutions u_1 and u_2 of the semilinear parabolic equation

$$u_t - \Delta u = f(u)$$

in a bounded domain Ω, with the same Dirichlet boundary data but different initial conditions. Show that if initially $u_1(x,0) \leq u_2(x,0)$ for all $x \in \Omega$, then this property holds for all time: $u_1(x,t) \leq u_2(x,t)$ for all $x \in \Omega$ and all $t > 0$. [Hint: show that $w = u_2 - u_1$ solves an equation of the form $w_t - \Delta w = c(x,t)w$.]

(7) Let u solve the semilinear equation

$$u_t - \Delta u = f(u)$$

in a bounded domain Ω, with a Dirichlet boundary condition $u = 0$ at $\partial\Omega$. Suppose $u_t \geq 0$ initially (in other words, suppose $\Delta u_0 + f(u_0) \geq 0$, where u_0 is the initial condition). Show that $u_t \geq 0$ for all $x \in \Omega$ and all $t > 0$. [Hint: start by differentiating the equation in time, to get a PDE satisfied by u_t.]

(8) Consider the semilinear equation

$$u_t - \Delta u = u^3,$$

in a bounded domain $\Omega \subset R^n$, with Dirichlet boundary condition $u = 0$ at $\partial\Omega$ and initial condition $u(x,0) = u_0(x)$. Show that if

$$E[u_0] = \int_\Omega \left(\frac{1}{2}|\nabla u_0|^2 - \frac{1}{4}u_0^4\right)\, dx < 0$$

then the solution "blows up," i.e. a classical solution ceases to exist in finite time. [Hint: start by noting that $\frac{d}{dt}E[u(t)] \leq 0$. Then derive a relation linking $\frac{d}{dt}\int_\Omega u^2\, dx$ with $E[u(t)]$.]