PDE I – Supplementary Problems on topics covered Dec 3. Distributed 12/7/2013. I am *not* asking you to write up or turn in these problems. But I recommend working them, as a way to reinforce your understanding of the classification of 2nd order PDE's, and the notion of a characteristic or non-characteristic surface (material covered in the second half of the 12/3 lecture, corresponding to the Lecture 12 notes). Problem 7 concerns higher dimensions and higher-order equations – material at the end of the Lecture 12 notes that we'll discuss at the beginning of the 12/10 lecture.

(1) Consider Tricomi's equation

$$u_{yy} - yu_{xx} = 0.$$

Observe that it's hyperbolic for $y > 0$ and elliptic for $y < 0$. Find the characteristics in the hyperbolic region.

(2) The equation describing (isentropic, irrotational, steady, compressible) 2D flow is

$$(\rho(\nabla\phi)\phi_x)_x + (\rho(\nabla\phi)\phi_y)_y = 0,$$

with $\rho = \rho(\nabla\phi)$ determined by Bernoulli's law

$$\frac{1}{2}|\nabla\phi|^2 + \frac{1}{\gamma - 1}c^2(\rho) = H.$$

Here γ and H are constants, and

$$c^2(\rho) = A\gamma\rho^{\gamma-1}$$

where A is another constant. Show that its type is determined by the ratio

$$M = \frac{|\nabla\phi|}{c(\rho)},$$

which is called the Mach number: the PDE is hyperbolic ("supersonic") if $M > 1$ and elliptic ("subsonic") if $M < 1$.

(3) Consider Laplace's equation in 2D with Cauchy data on the unit circle:

$$u = f(\theta), \quad \frac{\partial u}{\partial r} = g(\theta) \quad \text{at } r = 1.$$

Suppose f and g are analytic in θ. What conclusion can you draw from the Cauchy-Kowalewsky theorem?

(4) Explain the following statement: solving the Cauchy problem for the 2D Laplace equation

$$u_{xx} + u_{yy} = 0, \quad u(0, y) = f(y), \quad u_x(0, y) = g(y)$$

by the Cauchy-Kowalevsky theorem amounts to finding the analytic continuation of $f'(y) + ig(y)$ to the complex plane $z = y - ix$ (in a neighborhood of the real axis $x = 0$) by using its power series expansion.

(5) You've probably heard the statement that "the Cauchy problem for Laplace's equation is ill-posed." Justify it by means of a suitable class of examples. (Choose the domain to make your life simple, e.g. a halfspace or a ball.)

(6) We noted in class that the line $t = 0$ is characteristic for the 1D heat equation $u_t = u_{xx}$. So the Cauchy-Kowalevsky theorem is not applicable. And indeed it's *not* true that analytic data leads to an analytic solution. Demonstrate this by showing that if the initial data are $u_0(x) = 1/(1 + x^2)$ at $t = 0$ then the solution is not analytic at $(0, 0)$. (Hint: assume there is an analytic solution, and find its Taylor expansion at $(0, 0)$. Show the resulting series does not have a positive radius of convergence. This problem is from Evans; he attributes it to Kowalevsky herself.)

(7) Some practice with higher dimensions and higher-order equations:

(a) What space-time hypersurfaces are characteristic for the 2D wave equation $u_{tt} = u_{xx} + u_{yy}$?

(b) Show that for the biharmonic equation

$$u_{xxxx} + 2u_{xxyy} + u_{yyyy} = 0$$

every surface is noncharacteristic. (For this reason, we consider the biharmonic equation to be elliptic.)