PDE I – Supplementary Problems on Hamilton-Jacobi equations.
Distributed 12/10/2013. The material covered in class in the final lecture, on 12/10/2013, will not be on the Final Exam. But you may still want to reinforce your understanding of Hamilton-Jacobi equations. Here are some problems to help with that.

(1) The Lecture 13 notes "derive" a solution formula for the final-value problem

$$u_t + \frac{1}{2}|\nabla u|^2 = 0 \text{ for } t < T, \text{ with } u(x,T) = g \text{ at } t = T,$$

namely

$$u(x,t) = \max_z \left\{ g(z) - \frac{|z-x|^2}{2(T-t)} \right\}. \tag{1}$$

(I put "derive" in quotes because our treatment of the optimal control problem was honest, but our derivation of the associated Hamilton-Jacobi equation was only formal.)

(a) Reversing time, give an associated solution formula for the initial value problem

$$u_\tau - \frac{1}{2}|\nabla u|^2 = 0 \text{ for } \tau > 0, \text{ with } u(x,0) = g \text{ at } \tau = 0.$$

(b) Changing the max to a min, give analogous solution formulas for the final-value problem

$$u_t - \frac{1}{2}|\nabla u|^2 = 0 \text{ for } t < T, \text{ with } u(x,T) = g \text{ at } t = T,$$

and for the initial-value problem

$$u_\tau + \frac{1}{2}|\nabla u|^2 = 0 \text{ for } \tau > 0, \text{ with } u(x,0) = g \text{ at } \tau = 0.$$

(2) Find an optimal control problem and a solution-formula analogous to (1) for which the Hamilton-Jacobi equation is

$$u_t + \frac{1}{4}|\nabla u|^4 = 0 \text{ for } t < T, \text{ with } u(x,T) = g \text{ at } t = T,$$

(3) We discussed in lecture that when $g = |x|$ in one space dimension, the solution formula (1) gives $u(x,t) = \frac{1}{2}(T-t) + |x|$. What happens when we change the final-time condition to

$$g = \begin{cases} \frac{1}{\epsilon}x^2 & \text{if } |x| \le \epsilon/2 \\ |x| - \frac{\epsilon}{4} & \text{if } |x| \ge \epsilon/2. \end{cases}$$

(This is a C^1 approximation to $|x|$.) Does the resulting solution have continuous derivatives, or does its graph still have a sharp valley?

(4) Consider the viscously-perturbed eikonal equation on a 1D interval:

$$\begin{aligned} 1 - |u_x| + \epsilon u_{xx} &= 0 \quad \text{for } -1 < x < 1 \\ u &= 0 \quad \text{at } x = \pm 1. \end{aligned}$$

Assume the solution is C^2. (This is true; extra challenge: can you justify it?).

(a) Show that $v = u_x$ has the form

$$\begin{aligned} v &= -1 + e^{-x/\epsilon} \qquad \text{for } 0 < x < 1 \\ v &= +1 - e^{x/\epsilon} \qquad \text{for } -1 < x < 0. \end{aligned}$$

(b) Integrate once to find a formula for u, and show that as $\epsilon \to 0$ it approaches $1 - |x|$.

[Hint for (a): any critical point of u must be a maximum, since $u_x = 0$ implies $u_{xx} < 0$. Therefore u has just one critical point. To the left of it $u_x \geq 0$; to the right $u_x \leq 0$.]

(5) This problem is a special case of the "linear-quadratic regulator" widely used in engineering applications. The state is $y(s) \in R^n$, and the control is $\alpha(s) \in R^n$. There is no pointwise restriction on the values of $\alpha(s)$. The evolution law is

$$dy/ds = Ay + \alpha(s), \quad y(t) = x,$$

for some constant matrix A, and the goal is to minimize

$$\int_t^T |y(s)|^2 + |\alpha(s)|^2 \, ds + |y(T)|^2.$$

(In words: we prefer $y = 0$ along the trajectory and at the final time, but we also prefer not to use too much control.)

(a) Consider the value function

$$u(x,t) = \min \left\{ \int_t^T |y(s)|^2 + |\alpha(s)|^2 \, ds + |y(T)|^2 \right\}$$

(where the minimum is over all controls $\alpha(s)$, and the trajectory $y(s)$ satisfies $y(t) = x$). What Hamilton-Jacobi equation does u (formally) solve? Explain further why we should expect the relation $\alpha(s) = -\frac{1}{2}\nabla u(y(s))$ to hold along optimal trajectories.

(b) Since the problem is quadratic, it's natural to guess that the value function $u(x,t)$ takes the form

$$u(x,t) = \langle K(t)x, x \rangle$$

for some symmetric $n \times n$ matrix-valued function $K(t)$. Show that this u solves the Hamilton-Jacobi-Bellman equation exactly if

$$\frac{dK}{dt} = K^2 - I - (K^T A + A^T K) \text{ for } t < T, \quad K(T) = I$$

where I is the $n \times n$ identity matrix. (Hint: two quadratic forms agree exactly if the associated symmetric matrices agree.)