PDE - Lecture 9, 11/12/2013

Today: linear wave eqn in $\mathbb{R}^2$ + $\mathbb{R}^3$; also just a bit about numerical schemes.

Focus first on 3D wave eqn

$$u_{tt} - \Delta u = 0 \quad \text{in } \mathbb{R}^3$$
$$\left.\begin{array}{l} u = f \\ u_t = g \end{array}\right\} \text{ at } t = 0$$

There's a soln formula that's almost as simple as the 1D case:

$$u(x,t) = \fint_{|y-x|=t} \left[f(y) + t g(y) + (y-x)\cdot\nabla f(y) \right] dy$$

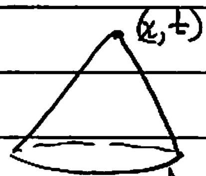
$$= \frac{1}{4\pi t^2} \int_{|y-x|=t} \left[\quad \right] dy$$

See eg

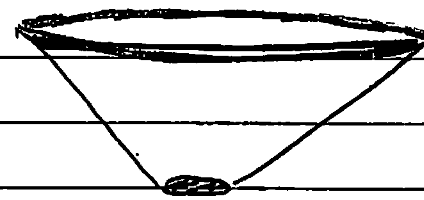
- John, §5.1 for a proof using the "method of spherical means" (sketched also below)
- Guenther + Lee, § 10.4 for a proof using something similar to a "fundamental soln"

Notice some key properties of soln:

a) True domain of dependence is a <u>sphere</u> not a ball: $u(x,t)$ depends only on data $(g, f, \nabla f)$ at y st $|y-x| = t$. (This is special to wave eqn in <u>odd</u> space dim $n=3,5,7$; note that $n=1$ is different!)



true domain of dependence is $\partial B_t(x)$ not $B_t(x)$



if supt of initial data is B_ρ, then its region of influence is a <u>shell</u> with thickness 2ρ

b) For compactly supported initial data, soln decays in amplitude like $1/t$ as $t \to \infty$

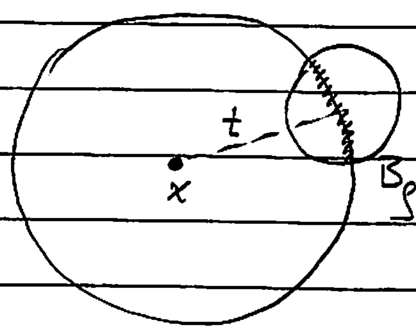
<u>Intuition</u>: energy is conserved $\left(\int u_t^2 + |\nabla u|^2 = \text{const}\right)$ but support spreads (it is confined to a growing shell) so amplitude should decay.

Justification: suppose spt of $f+g \subset B_\rho$. Recall

$$u = \fint_{|y-x|=t} [f + tg + (y-x)\cdot\nabla f]$$

At time t, integrand vanishes except on

$\{y : |y-x|=t \text{ \& } y \in B_\rho\}$, which has 2D area $\sim \rho^2$ if t is large



and on its spt the integrand is of order t. So u is at most

$$\underbrace{\frac{\text{const}}{t^2}}_{\text{normalization}} \cdot \underbrace{t}_{\text{integrand value}} \cdot \underbrace{\rho^2}_{\text{area}} \sim \frac{C\rho^2}{t}$$

[Note that 1D wave eqn is different: solns with compactly spt'd data don't decay in magnitude.]

c) Soln of 3D wave eqn is in general less regular than initial data (since soln formula involves ∇f as well as f)

d) essential features of soln are evident even in radial setting, since soln formula tells us that at $x=0$,

$$\underbrace{u(0,t) = \fint_{|y|=t} [f + tg + y\cdot\nabla f]}_{\text{feels only the averages of } f+g \text{ on spheres of various radii}}$$

Soln to 2D wave eqn

$$u_{tt} - \Delta u = 0 \quad \text{in } \mathbb{R}^2$$
$$\left.\begin{aligned} u &= f \\ u_t &= g \end{aligned}\right\} \text{ at } t=0$$

can be deduced from the 3D formula (and a similar calcn can be done in any even dim $n=2k$, starting from soln formula in dim $2k+1$). Strategy is simple ("method of descent"): extend the initial data to $\mathbb{R}^3$, as fns indep of x_3, then solve in $\mathbb{R}^3$, then evaluate soln at $(x_1, x_2, 0; t)$.

This calcn is most easily applied to a reorganized version of 3D soln formula, namely

(3D) $$u(x,t) = t \fint_{|y-x|=t} g + \frac{d}{dt}\left[t \fint_{|y-x|=t} f \right]$$

[See pg 9.13 for pf that this is equiv to 3D formula given earlier.] The outcome is

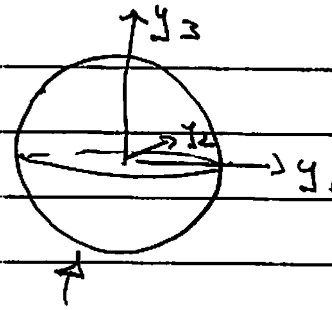
(2D) $$u(x_1, x_2; t) = \frac{1}{2\pi} \int_{|y-x|<t} \frac{g(y)}{(t^2-|y-x|^2)^{1/2}}\, dy_1\, dy_2$$

$$+ \frac{d}{dt}\left[\frac{1}{2\pi} \int_{|y-x|<t} \frac{f(y)}{(t^2-|y-x|^2)^{1/2}}\, dy_1\, dy_2 \right]$$

Expln why: the "3D spherical mean"

$$\frac{1}{4\pi t^2} \int_{|y_1-x_1|^2+|y_2-x_2|^2+y_3^2} f(y_1, y_2)\, d(\text{area})$$

can be written as an integral wrt (y_1, y_2):



$$y_3^2 = t^2 - |y_1 - x_1|^2 - |y_2 - x_2|^2$$

center $= (x_1, x_2, 0)$

$$d(\text{area}) = \left[1 + \left(\frac{\partial y_3}{\partial y_1}\right)^2 + \left(\frac{\partial y_3}{\partial y_2}\right)^2\right]^{1/2} dy_1\, dy_2$$

$$= \frac{t}{|y_3|}\, dy_1\, dy_2$$

The prefactors in the 2D formula are $\frac{1}{2\pi}$ not $\frac{1}{4\pi}$ since the upper + lower hemispheres both contribute equally in the preceding calcn.

A key property of 2D soln formula:

- in 2D, domain of dependence is the *entire* *ball* $|y - x| < t$. So: in 2D a localized disturbance takes the expected time to reach an observer (dictated by the wave speed) but thereafter the source will "continue to be seen forever" (albeit with decaying amplitude).

What about numerical solutions? Simplest approach – one that works even for *nonlinear wave eqns* – is to discretize space but keep time conts. Discrete scheme should *maintain*

The "Hamiltonian" structure, ie should have

$$\frac{d}{dt}\,(\text{kinetic} + \text{potential energy}) = 0$$

for a suitable discretized version of the "potential energy". For example, in 1D on $[0, L]$ with Dirichlet condition $u = 0$ at ends $x = 0, L$, consider nonlin wave eqn

$$u_{tt} - u_{xx} + u^3 = 0 .$$

Assoc cons law is

$$\frac{d}{dt}\left(\int_0^L \tfrac{1}{2}u_t^2 + \tfrac{1}{2}u_x^2 + \tfrac{1}{4}u^4\,dx\right) = 0$$

Discretizing in space, we should solve for

$$u_j(t) \simeq u(j\Delta x, t) \qquad \text{where } \Delta x = \frac{L}{N}$$

We then want

$$\frac{d}{dt}\left(\sum_j{}' \tfrac{1}{2}\dot{u}_j^2 + \frac{|u_j - u_{j-1}|^2}{2(\Delta x)^2} + \tfrac{1}{4}u_j^4\right) = 0$$

which we can achieve by solving the ODE system

$$\ddot{u}_j = \frac{u_{j-1} + u_{j+1} - 2u_j}{(\Delta x)^2} - u_j^3$$

(The bc $u_0 = 0$, $u_N = 0$ are needed to get $\ddot{u}_1$ and $\ddot{u}_{N-1}$; this ODE needs initial conds for $u_j(0)$ + $\dot{u}_j(0)$ as expected).

What about a time-discrete scheme? The simplest (explicit) method for lin wave eqn is

$$\frac{u_j(t+\Delta t) + u_j(t-\Delta t) - 2u_j(t)}{(\Delta t)^2} = \frac{u_{j+1}(t) + u_{j-1}(t) - 2u_j(t)}{(\Delta x)^2}$$

which reorganizes to

$$u_j(t+\Delta t) = -u_j(t-\Delta t) + \left(\frac{\Delta t}{\Delta x}\right)^2 \left[u_{j+1}(t) + u_{j-1}(t) \right] + 2\left(1 - \left(\frac{\Delta t}{\Delta x}\right)^2\right) u_j(t)$$

Fact: this scheme is unstable if $(\Delta t/\Delta x)^2 > 1$, but it is stable if $(\Delta t/\Delta x)^2 < 1$. (See eg Strauss 2.8.3.) Heuristic interpretation: in discrete scheme info can propagate at most one unit Δx at each time step Δt, ie at velocity $\Delta x/\Delta t$. So of course we expect a reqt of form $\Delta x/\Delta t > 1$ for the numerical scheme to get a correct approximate soln.

The rest of these notes derive the 3D soln formula using the method of spherical means (essentially the same as you'll find in John).

<u>Notation</u>: if $\varphi = \varphi(x)$ is defined on $\mathbb{R}^n$ then

$M_\varphi(x,r)$ = avg of φ on <u>sphere</u> of rad r centered at x

$$= ⨍_{|y-x|=r} \varphi(y) = \frac{1}{n\alpha(n)r^{n-1}} \int_{\partial B_r(x)} \varphi$$

[Note: if $\alpha(n) r^n$ is vol of B_r, then $n\alpha(n)r^{n-1}$ is area of ∂B_r]. We may take the convention

$$M_\varphi(x,r) = M_\varphi(x,-r) \quad \text{for } r<0;$$

Then

$$M_\varphi(x,r) = \frac{1}{n\alpha(n)} \int_{|y|=1} \varphi(x+ry)\,dy$$

holds even for $r<0$.

<u>Claim</u>: If u solves wave eqn in $\mathbb{R}^n$, then for any fixed x, M_u solves the pde in $r+t$:

$$\partial_{rr} M_u + \frac{n-1}{r}\partial_r M_u = \partial_{tt} M_u \qquad (t>0).$$

[Note: This is just the wave eqn in radial coords in $\mathbb{R}^n$, since if $\varphi = \varphi(r)$ in $\mathbb{R}^n$ then $\Delta\varphi = \varphi_{rr} + \frac{n-1}{r}\varphi_r$. A painless way to see this is to consider the 1st var of $\int |\nabla\varphi|^2 dx = c\int |\varphi'(r)|^2 r^{n-1} dr$.]

Proof of claim by direct calcn:

Observe first that for any φ,

$$(\ast\ast) \qquad \partial_r M_\varphi = \frac{r}{n} ⨍_{B_r(x)} \Delta\varphi$$

since

$$\partial_r M_\varphi = \frac{1}{n\alpha(n)} \partial_r \int_{|y|=1} \varphi(x+ry)\, dy$$

$$= \frac{1}{n\alpha(n) r^{n-1}} \int_{\partial B_r(x)} \frac{\partial\varphi}{\partial n}$$

$$= \frac{1}{n\alpha(n) r^{n-1}} \int_{B_r(x)} \Delta\varphi$$

$$= \frac{r}{n} \frac{1}{\alpha(n) r^n} \int_{B_r(x)} \Delta\varphi$$

[Note as a consequence of the above that $\partial_r M_\varphi \to 0$ as $r \to 0$ if φ is C^2. Also, differentiating in r,

$$\partial_{rr} M_\varphi = -(n-1) r^{-n} \frac{1}{n\alpha(n)} \int_{B_r(x)} \Delta\varphi$$

$$+ \frac{1}{n\alpha(n) r^{n-1}} \int_{\partial B_r(x)} \Delta\varphi$$

$$= -\frac{(n-1)}{n} ⨍_{B_r(x)} \Delta\varphi + ⨍_{\partial B_r(x)} \Delta\varphi$$

so $\lim_{r\to 0} \partial_{rr} M_\varphi = \frac{1}{n} \Delta\varphi(x)$ exists if φ is C^2.]

Continuing the pf of the claim: if u solves wave eqn then (**) becomes

$$\partial_r M_u = \frac{1}{n\alpha(n) r^{n-1}} \int_{B_r(x)} u_{tt}$$

so

$$r^{n-1} \partial_r M_u = \frac{1}{n\alpha(n)} \int_0^r d\rho \int_{|y-x|=\rho} u_{tt}.$$

$$\Rightarrow \quad \partial_r \left(r^{n-1} \partial_r M_u \right) = \frac{1}{n\alpha(n)} \int_{|y-x|=r} u_{tt}$$

$$= r^{n-1} \partial_{tt} M_u$$

Thus $$\partial_{tt} M_u = \frac{1}{r^{n-1}} \partial_r \left(r^{n-1} \partial_r M_u \right)$$

which is equiv to (*).

Now we deduce the 3D soln formula.
When $n=3$, the claim says

$$(r M_u)_{tt} = (r M_u)_{rr}$$

so $r M_u$ solves the 1D wave eqn in r + t.
By 1D soln formula,

$$r M_u(x, r; t) = \frac{1}{2}\left[(r+t) M_f(x; r+t) + (r-t) M_f(x; r-t) \right]$$
$$+ \frac{1}{2}\int_{r-t}^{r+t} \xi M_g(x, \xi)\, d\xi$$

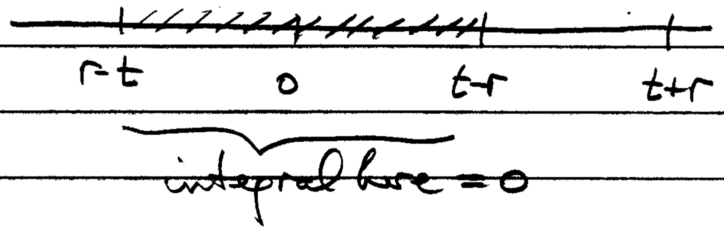
since the initial data are

$$f(r) = r M_u \Big|_{t=0} = r M_f(x, r)$$

$$g(r) = \partial_t r M_u \Big|_{t=0} = r M_g(x, r)$$

Since M_g is <u>even</u> in r, we can rewrite this (for $r<t$) as

$$rM_u(x;r,t) = \tfrac{1}{2}\Big[(t+r)M_f(x,r+t) - (t-r)M_f(x,t-r)\Big] + \tfrac{1}{2}\int_{t-r}^{t+r} \xi M_g(x,\xi)\,d\xi.$$



Taking the limit $r \to 0$ we get

$$u(x,t) = \partial_t\Big[tM_f(x,t)\Big] + tM_g(x,t)$$

which is the form of the soln formula we used on pg 9.5. To get the the other version (stated on pg 9.1) observe that

$$\partial_t M_f(x,t) = \partial_t \fint_{|z|=1} f(x+tz)\,dz$$

$$= \fint_{|z|=1} z\cdot\nabla f(x+tz)\,dz$$

$$= \fint_{|y-x|=t} \left(\frac{y-x}{t}\right)\cdot\nabla f\,dy$$

so

$$u(x,t) = \fint_{|y-x|=t} \left[f + tg + (y-x)\cdot\nabla f \right] dy$$

as expected.