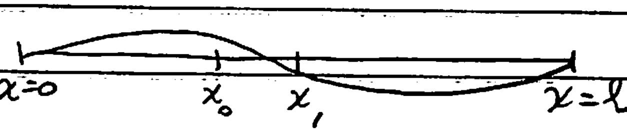
PDE - Lecture 8, 11/5/2013

Today + next wk: linear wave eqn $u_{tt} - \Delta u = 0$ + related matters.

Some examples of probs leading to this eqn:

A) vibrating string ($u_{tt} - c^2 u_{xx} = 0$) or vibrating membrane ($u_{tt} = c^2(u_{xx} + u_{yy})$).

Expln, following Strauss (see also Guenther + Lee §1.2): consider elastic string of length l undergoing transverse vibrations (eg violin string); assume it's under tension $T(x,t)$. Set ρ = density (const), + let $u(x,t)$ = vert displacement.



Look at Newton's law "force = mass × acceleration" for the part of string btwn x_0 + x_1. Noting that slope of string is u_x, we get

$$\left. \frac{T}{\sqrt{1+u_x^2}} \right|_{x_0}^{x_1} = 0$$

LHS is net horizontal force; RHS = 0 since there's no horizontal motion

$$\left. \frac{T u_x}{\sqrt{1+u_x^2}} \right|_{x_0}^{x_1} = \int_{x_0}^{x_1} \rho u_{tt} \, dx$$

LHS is net vertical force; RHS is vert component of accel.

Now assume slope is small; then

$$(1+u_x^2)^{1/2} \simeq 1 + \tfrac{1}{2}u_x^2$$

At leading order, 1st eqn says T is indep of x. We then expect that it's also indep of t, + 2nd eqn becomes

$$T u_{xx} = \rho u_{tt}$$

i.e. $u_{tt} = c^2 u_{xx}$, $c = \sqrt{T/\rho}$ (as we'll see, c = <u>wave speed</u>).

Variations on this:

- <u>air resistance</u> introduces additional "damping" force in vertical eqn, leading to $\rho u_{tt} - T u_{xx} + \gamma u_t = 0$

Note: if eg $u=0$ at ends then

$$\frac{d}{dt}\int_0^\ell \frac{\rho}{2}u_t^2 + \frac{T}{2}u_x^2 = \int_0^\ell \rho u_t u_{tt} + T u_x u_{xt}$$

$$= \int_0^\ell u_t(\rho u_{tt} - T u_{xx})$$

$$= -\gamma \int_0^\ell u_t^2\, dx$$

which is indeed a damping term if $\gamma > 0$.

[$\int_0^\ell \frac{\rho}{2}u_t^2 + \frac{T}{2}u_x^2\, dx$ = kinetic + potential energy!]

- 2D case is very similar, provided state of stress is uniform tension T.

For any region $D \subset$ membrane,

$$\underset{\text{on } D}{\text{vert force}} = \int_{\partial D} T \frac{\partial u}{\partial n}\, ds = \int_D \rho u_{tt}$$

Therefore

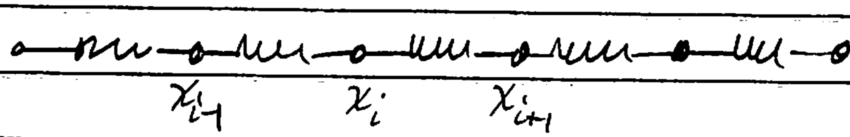
$$\int_D \operatorname{div}(T \nabla u) = \int_D \rho u_{tt} \quad \text{for any region } D$$

so (recalling hypoth of const T)

$$\rho u_{tt} - T \Delta u = 0$$

i.e. $\quad u_{tt} = c^2 \Delta u \quad \left(c = \sqrt{T/\rho} \text{ as before}\right).$

(B) Longitudinal vibration of linear springs leads to a finite-difference version of wave eqn



This time let u_i = displacement left or right of x_i; assume spring constant E & mass ρ (the same for each spring + node). Then

Net force on node i = $E(u_{i+1}-u_i) - E(u_i - u_{i-1})$.

accel of node i = $\ddot{u}_i = \partial^2_{tt} u_i$

So force = mass x accel implies

$$\rho \ddot{u}_i = = E(u_{i+1} + u_{i-1} - 2u_i)$$

Note: for example (A) we got a linear eqn because we took a small-slope approxn; in example (B) the eqn is linear with no approx (but it's easy to introduce nonlinearity by considering nonlinear springs).

(C) Maxwell's eqns describe evolution of electromagnetic fields. In simplest setting (no current, no charges) they say

$$\frac{\varepsilon}{c} E_t = \text{curl } H \qquad \text{div } H = 0$$

$$\frac{\mu}{c} H_t = -\text{curl } E \qquad \text{div } E = 0.$$

From vector identity curl curl $H = \nabla \text{div } H - \Delta H$ we get, if ε + c are constant,

$$\frac{\varepsilon}{c} (\text{curl } E)_t = -\Delta H \implies \frac{\varepsilon\mu}{c^2} H_{tt} = \Delta H.$$

Similarly (if μ is also constant)

$$\frac{\mu}{c}(\mathrm{curl}\, H)_t = \Delta E \Rightarrow \frac{\varepsilon\mu}{c^2} E_{tt} = \Delta E$$

So each component of $H + E$ solves a scalar wave eqn with wavespeed $\frac{c}{\sqrt{\varepsilon\mu}}$

There are plenty more examples (eg acoustics), but it's time to discuss solns. We focus for rest of this lecture on the case of one space dimension, which is already nontrivial (eg vibrating string) but relatively easy.

Focus first on Cauchy problem (with $c=1$ for simplicity):

$$u_{tt} - u_{xx} = 0 \qquad x \in \mathbb{R}, \quad t > 0$$

$$\left.\begin{aligned} u(x,0) &= f(x) \\ u_t(x,0) &= g(x) \end{aligned}\right\} \text{ at } t=0$$

Notes: a) unlike heat eqn we must specify both $u + u_t$ at $t=0$

b) sign in eqn is crucial: $u_{tt} - u_{xx} = 0$ is wave eqn, while $u_{tt} + u_{xx} = 0$ would be a 2D Laplace eqn (and we wouldn't be able to specify both $u + u_t$!).

General 1D soln formula:

$$u = F(x+t) + G(x-t)$$

Easy to check that *if* u has this form *then* it solves eqn. For converse: let $\xi = x+t$, $\eta = x-t$. Then $x = \frac{1}{2}(\xi+\eta)$, $t = \frac{1}{2}(\xi-\eta)$, so $\varphi_\xi = \varphi_x \frac{\partial x}{\partial \xi} + \varphi_t \frac{\partial t}{\partial \xi}$ $\Rightarrow$

$$\varphi_\xi = \tfrac{1}{2}(\varphi_x + \varphi_t)$$
$$\varphi_\eta = \tfrac{1}{2}(\varphi_x - \varphi_t)$$

Thus $\quad u_{tt} - u_{xx} = 0 \iff (\partial_t - \partial_x)(\partial_t + \partial_x) u = 0$

$$\iff -4\partial_{\xi\eta} u = 0.$$

Now, $\partial_{\xi\eta} u = 0 \Rightarrow u = F(\xi) + G(\eta)$.

In fact, it's easy to read off F + G from the initial data: at $t=0$ evidently

$$u(x,0) = F(x) + G(x) = f(x)$$
$$u_t(x,0) = F'(x) - G'(x) = g(x)$$

so $\quad \begin{aligned} F' + G' &= f' \\ F' - G' &= g \end{aligned} \quad \xRightarrow[\text{2x2 system}]{\text{solving the}} \quad F' = \dfrac{f'+g}{2}, \quad G' = \dfrac{f'-g}{2}$

Integrating:

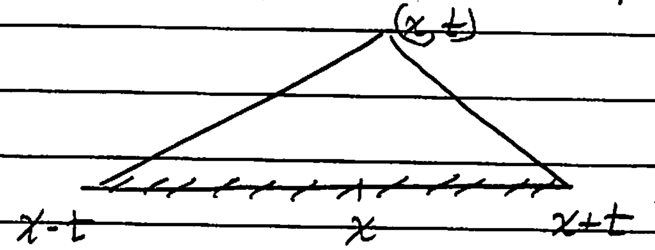
$$F(x) = \tfrac{1}{2} f(x) + \tfrac{1}{2}\int_0^x g(\xi)\,d\xi + c_0$$

$$G(x) = \tfrac{1}{2} f(x) - \tfrac{1}{2}\int_0^x g(\xi)\,d\xi - c_0$$

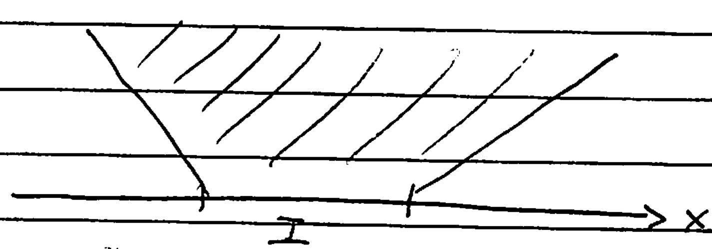
where c_0 is a constant of integr. (There's only one const of integr since $F+G = f$.) We can set $c_0 = 0$ since its value doesn't affect u. Finally:

$$\boxed{\begin{aligned} u(x,t) &= F(x+t) + G(x-t) \\ &= \tfrac{1}{2}\left[f(x+t) + f(x-t)\right] + \tfrac{1}{2}\int_{x-t}^{x+t} g(\xi)\,d\xi \end{aligned}}$$

Note phenomenon of domain of dependence:



$u(x,t)$ depends on the initial data only in the interval $(x-t, x+t)$



data in I at $t=0$ influence soln only in the shaded region.

Conclusions + notes:

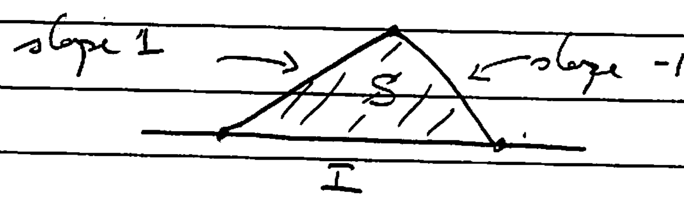
- "information propagates at finite speed" (speed 1, for $u_{tt} - u_{xx} = 0$). Very different from heat eqn!

- eqn does not smooth its initial data

- eqn can be solved backward in time just as easily as forward in time

For eqn $u_{tt} - c^2 u_{xx} = 0$ story is essentially the same (of course), except that $u = F(x+ct) + G(x-ct)$. Information propagates at speed c.

There's an alternative "energy-based" pf of domain of dependence, which is important because it extends straightforwardly to higher dims (where exact soln formulas exist but are more complicated). Focus as before on $c=1$ ($u_{tt} - u_{xx} = 0$).



Claim: if $u = u_t = 0$ on I and $u_{tt} - u_{xx} = 0$ then $u \equiv 0$ everywhere in S (see figure).

Pf: May suppose (by replacing u with $u_\lambda(x,t) = u(\lambda x, \lambda t)$) that $I = (-1,+1)$. Consider the "energy" in each time slice

$$e(t) = \int_{-1+t}^{1-t} \tfrac{1}{2}u_t^2 + \tfrac{1}{2}u_x^2 \, dx$$

$$\frac{de}{dt} = \int_{-1+t}^{1-t} u_t u_{tt} + u_x u_{xt} \, dx + \text{bdry terms}$$

$$= \int_{-1+t}^{1-t} u_t \underbrace{(u_{tt} - u_{xx})}_{0} \, dx + \text{different bdry terms}$$

We could proceed to find bdry terms using Calc I; but it's more efficient to <u>start over again</u>, looking for an appropriate "integr by parts".

Consider vector field in (x,t) plane:

$$\sigma = \left[-u_x u_t, \ \tfrac{1}{2}u_t^2 + \tfrac{1}{2}u_x^2 \right]$$

Note that $\operatorname{div} \sigma = \partial_t \left[\tfrac{1}{2}u_t^2 + \tfrac{1}{2}u_x^2 \right] + \partial_x \left[-u_x u_t \right]$

$$= u_t u_{tt} + \cancel{u_x u_{xt}} - u_t u_{xx} - \cancel{u_x u_{tx}}$$

$$= u_t (u_{tt} - u_{xx})$$

[This is precisely what we used earlier!]. So

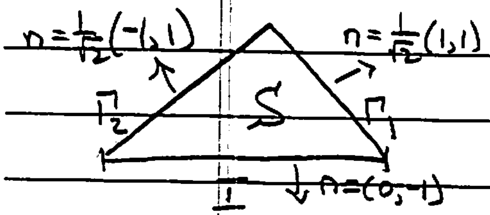
$u_{tt} - u_{xx} = 0$ in S $\Rightarrow$

$$0 = \int_S \operatorname{div} \sigma = \int_{\partial S} \sigma \cdot n$$

$$= -\int_I \tfrac{1}{2}(u_t^2 + u_x^2)$$

$$+ \frac{1}{\sqrt{2}} \int_{\Gamma_1} \left(\tfrac{1}{2} u_t^2 + \tfrac{1}{2} u_x^2 - u_x u_t\right) ds$$

$$+ \frac{1}{\sqrt{2}} \int_{\Gamma_2} \left(\tfrac{1}{2} u_t^2 + \tfrac{1}{2} u_x^2 + u_x u_t\right) ds$$

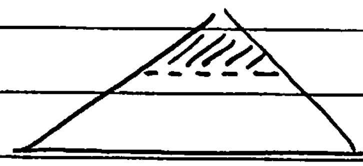


On $\Gamma_1 + \Gamma_2$ the integrands are ≥ 0. So

$$u_t = u_x = 0 \text{ on } I \Rightarrow \int_{\Gamma_1} (u_t - u_x)^2 + \int_{\Gamma_2} (u_t + u_x)^2 = 0$$

$$\Rightarrow u_t = u_x \text{ on } \Gamma_1$$
$$u_t = -u_x \text{ on } \Gamma_2$$

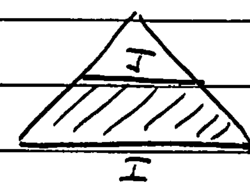
Now applying same calcn on subtriangles



we see that $\int_{-1+t}^{1-t} u_t^2 + u_x^2 \, dx = 0$ for any $0<t<1$.

So $u_x \equiv 0$ and $u_t \equiv 0$ throughout S. Hence (using initial data again) $u \equiv 0$ in S.

[A slightly better way to organize this arg't: integrate div σ not over all S, but rather over the shaded region



to get

$$0 = \frac{1}{2}\int_J u_t^2 + u_x^2 - \frac{1}{2}\int_I u_t^2 + u_x^2 + [\text{nonneg terms along left + right bdries}].$$

This gives not only the result we proved above but also the estimate, for <u>any</u> choice of initial data, that

$$\frac{1}{2}\int_J u_t^2 + u_x^2 \le \frac{1}{2}\int_I u_t^2 + u_x^2 \, . \,]$$

What about bounded domains, eg

$$u_{tt} - u_{xx} = 0 \quad \text{for} \quad 0 < x < L$$

with bc $u = 0$ at $x = 0, L$, or else $u_x = 0$ at $x = 0, L$?
Sep of vars is a good tool here, using eigenfns of Laplacian with the chosen bc:

$$u = \sum a_j(t)\varphi_j(x) \qquad \text{where} \quad -\Delta\varphi_j = \lambda_j\varphi_j$$

(using homog Dir bc $\varphi_j = 0$ if $u = 0$ at bdry, or homog Neumann bc $\partial\varphi_j/\partial n = 0$ if $\partial u/\partial n = 0$). Note: this method works the same in higher dim's as in 1D. Coeffts must solve $\ddot{a}_j = -\lambda_j a_j$, so

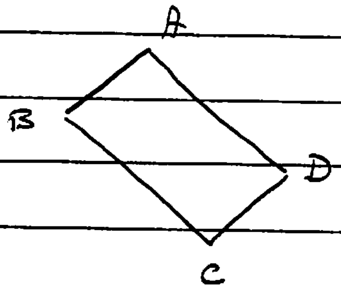
$$a_j(t) = \alpha_j \cos\sqrt{\lambda_j}\, t + \beta_j \sin\sqrt{\lambda_j}\, t$$

[or, in complex notation, $a_j = \mathrm{Re}\left(c_j e^{i\sqrt{\lambda_j}\, t}\right)$].
Initial data determine α_j and β_j.

A similar repn is possible for $u_{tt} = \Delta u$ in all $\mathbb{R}^n$ using Fourier transform.

But: sep of vars soln hides the fact that information propagates at speed 1. To capture this fact in 1D, when $u = 0$ at bdry, we can proceed differently, as follows:

__First__: observe that for a parallelogram with sides of slope ± 1 in space time



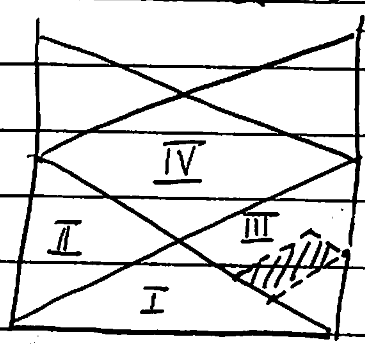
$$u_{tt} - u_{xx} = 0 \text{ inside } \Rightarrow u(A) - u(B) - u(D) + u(C) = 0.$$

__Pf__: recall that eqn says $u_{\xi\eta} = 0$ when $\xi = x+t$, $\eta = x-t$, and our parallelogram is a rectangle (aligned with the axes) in ξ,η plane. Now apply elementary calculus.

__Now apply this__: if

$$u_{tt} - u_{xx} = 0 \qquad 0 < x < L$$
$$u = 0 \quad \text{at } x = 0, L$$
$$\left.\begin{array}{l} u = f(x) \\ u_t = g(x) \end{array}\right\} \text{ at } t = 0$$

we can determine u iteratively



- in region I – bdry has no effect
- in regions II + III – use bdry for one vertex of parallelogram
- etc

For a half-line, of course we can use reflection. (Also for an interval!). In fact:

- to solve $u_{tt} - u_{xx} = 0$ on $x>0$, $t>0$ with bc $u=0$ at $x=0$, look for a soln of $u_{tt}-u_{xx}=0$ on all $\mathbb{R}$ that's <u>odd</u> in x, by using odd reflection of initial data
- to solve $u_{tt}-u_{xx}=0$ on $x>0$, $t>0$ with bc $u_x=0$ at $x=0$, look for soln on all $\mathbb{R}$ that's <u>even</u> in x, by using even extnsn of initial data
- to solve $u_{tt}-u_{xx}=0$ on $0<x<L$, $t>0$ with $u=0$ (or $u_x=0$) at $x=0, L$, use odd (or even) reflection to create $2L$-periodic initial data on real line then use soln formula for real line.

(These are, of course, tricks we also used for the heat eqn.)

What if the RHS is not zero? Let's discuss

$$(*) \quad \begin{aligned} u_{tt} - \Delta u &= w(x,t) \quad \text{in } \mathbb{R}^n, \text{ for } t>0 \\ u &= f \\ u_t &= g \end{aligned} \Big\} \text{ at } t=0$$

(The following discn is really independent of spatial dimension.)

Like the heat eqn, we can view wave eqn as an "ODE in function space", eg $u_{tt} = \Delta u$ iff

$$\frac{d}{dt}\begin{pmatrix} u \\ u_t \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ \Delta & 0 \end{pmatrix}\begin{pmatrix} u \\ u_t \end{pmatrix}$$

so we can use something like the "variation of parameters formula" to represent the soln with nonzero w.

But the "ODE framework" is perhaps more confusing than useful, so let's skip straight to the answer: to solve $(*)$, we look for

$$u = u_1 + u_2$$

where

u_1 solves the homogeneous pbm
($w=0$; $f+g$ as given)

u_2 solves the inhomogeneous pbm with zero initial data ($u_2 = \partial_t u_2 = 0$ at $t=0$).

Formula for u_1 is already discussed in $\mathbb{R}^1$; we'll discuss higher dimensions soon.

Formula for u_2 is

$$u_2(x,t) = \int_0^t U(x,t;s)\,ds$$

where for fixed s, $U(x,t;s) \equiv 0$ for $t<s$ and it solves pde

$$U_{tt} - \Delta U = 0 \qquad t>s$$
$$U(x,s;s) = 0$$
$$U_t(x,s;s) = w(x,s)$$

We can check that this works by direct calcu:

$$\partial_t u_2 = \overbrace{\cancel{U(x,t;t)}}^{0} + \int_0^t U_t(x,t;s)\,ds$$

$$\partial_{tt} u_2 = U_t(x,t;t) + \int_0^t U_{tt}(x,t;s)\,ds$$

$$= w(x,t) + \int_0^t \Delta U(x,t;s)\,ds$$

while $\quad \Delta u_2 = \int_0^t \Delta U(x,t;s)\,ds$

so $\quad (\partial_{tt} - \Delta)\, u_2 = w$. Also $u_2 = \partial_t u_2 = 0$ at $t=0$,

as desired.