PDE - Lecture 7, 10/29/2013

One more lecture on Laplace's eqn + related topics.

a) Some topics addressed in Lect 6 notes but not yet covered in class

- representing solns to Neumann prbms ($-\Delta u = f$ in Ω, $\partial u/\partial n = g$ at $\partial\Omega$) using the "Neumann function"

- regularity, viewed from perspective of Poisson integral formula: if $\Delta u = 0$ in B_r then

$$\boxed{u(x) = \int_{|y|=r} K(x,y)\, u(y)\, dA_y} \quad (*)$$

$$K(x,y) = \frac{1}{r\,|\partial B_1|} \frac{r^2 - |x|^2}{|y-x|^n} \quad \text{in } \mathbb{R}^n$$

[discussed on pg 9 of Lecture 6, but the formula for K was missing + there was a "typo" in the version of (*) written there].

* regularity, viewed from perspective of Harnack's inequality

b) Brief overview of some methods for proving existence of solutions and/or finding solutions numerically

c) Variational method, both as a way of

characterizing solns and as a way of finding them numerically.

See end of these notes for suggestions for add'l reading (topics we might have covered but didn't have time for).

The "big picture" on existence. (Note: it isn't suff't to "just use the Green's fn," since our arg't for the existence of the Green's fn assumed that you can solve $-\Delta u = 0$ in Ω, $u|_{\partial\Omega} = g$ when g is smooth!)

There are 3 main approaches:

(1) Perron's method for solving $\Delta u = 0$ in Ω, $u = g$ at $\partial\Omega$ rests on 2 observations:

- if $\Delta w \geq 0$ in Ω ("w is subharmonic") and $w \leq u$ at $\partial\Omega$ then $w \leq u$ in Ω (by max prin, since $\Delta(w-u) \geq 0$)

- same stmt even if w is merely cont's and "$\Delta w \geq 0$" is replaced by

$$u(x) \leq \frac{1}{|\partial B_r(x)|} \int_{|y-x|=r} u(y)\, dA_y$$

(since pf of max prin uses only mean value formula)

So: soln can be characterized as

u = largest cont's fn w s.t. $w \le g$ at $\partial\Omega$ and w is subharmonic (in sense defined by 2nd bullet).

See F. John §4.4 for pf that there is such a fn, and that it is harmonic, with $u = g$ at $\partial\Omega$.

Advantage of this approach: it permits very general bc and rather singular domains (though some hypoth is needed!)

Disadvantage: not well-suited to numerical implementation.

(2) Boundary integral method: solve an integral eqn on $\partial\Omega$, for example solving $\Delta u = 0$ in Ω, $u = g$ at $\partial\Omega$ by seeking a fn $\varphi : \partial\Omega \to \mathbb{R}$ s.t. the fn u_φ defined (in $\mathbb{R}^n$, $n \ge 3$) by

$$u_\varphi(x) = \int_{\partial\Omega} \frac{\partial}{\partial \nu_y}\left(\frac{1}{|x-y|^{n-2}}\right) \varphi(y)\, dA_y \qquad (**)$$

(which is definitely harmonic in Ω) has the desired bc g at $\partial\Omega$. (This is called the rep'n of u via a "double layer potential".)

See Guenther + Lee §8.6 – 8.7 for a good treatment of this topic.

Advantage: works well numerically (it's the method of choice for unbounded domains, where discretizing space is impractical).

Disadvantage: method is rather special to Laplace's eqn, + relies heavily on the linearity of the eqn.

(3) Variational principles, eg

a) $\begin{array}{l} \Delta u = f \text{ in } \Omega \\ u = g \text{ at } \partial\Omega \end{array} \iff \min\limits_{u|_{\partial\Omega} = g} \int_\Omega \frac{1}{2}|\nabla u|^2 + fu$

b) $\begin{array}{l} \Delta u = f \text{ in } \Omega \\ \frac{\partial u}{\partial n} = g \text{ at } \partial\Omega \end{array} \iff \min\limits_{u} \int_\Omega \left(\frac{1}{2}|\nabla u|^2 + fu\right) dx - \int_{\partial\Omega} ug \, dA$

Advantages: easy to implement numerically (by minimizing over a finite-dim'l class of u's), and extends easily to many nonlinear plms.

(Note: not every pde comes from a var'l principle. But for linear eqns there's a fix — known as "Lax Milgram lemma" — that permits even eqns not assoc to var'l prin's to be reduced to lin alg plm similar to the one assoc with a var'l prin.)

Impt comments on (a) + (b)

- In (b) it is crucial that the data be consistent; otherwise the min is $-\infty$, achieved by taking $u = \text{const}$ and letting the const go to $\pm\infty$.

- The fact that (a) has 1st varn 0 at the soln of the pde is familiar by now. For (b), this seen as follows:

$$\frac{d}{dt}(\text{value of fnl at } u + t\varphi) = 0 \quad \text{at } t = 0 \implies$$

$$\int_{\partial\Omega}\left(\frac{\partial u}{\partial n} - g\right)\varphi + \int_{\Omega}(-\Delta u + f)\varphi = 0 \quad \text{for all } \varphi.$$

Taking $\varphi = 0$ near $\partial\Omega$, we argue as usual that this $\Rightarrow -\Delta u + f = 0$ in Ω.

Since there is no bc on φ, and we now know that $\int_{\partial\Omega} \left(\frac{\partial u}{\partial n} - g\right)\varphi = 0$ for all φ, we conclude as usual that $\frac{\partial u}{\partial n} - g = 0$ at $\partial\Omega$.

∘ To see that the functional is minimized at u, we use its convexity — or, in this case, its quadratic + linear character.

For (a): Let u solve the pde, and let w have the same bc g. Then writing

$$\frac{1}{2}|\nabla w|^2 = \frac{1}{2}|(\nabla u) + \nabla(w-u)|^2$$

$$= \frac{1}{2}|\nabla u|^2 + \langle \nabla u, \nabla(w-u)\rangle + \frac{1}{2}|\nabla(w-u)|^2$$

we find that

$$\int_\Omega \frac{1}{2}|\nabla w|^2 + f w = \int_\Omega \frac{1}{2}|\nabla u|^2 + f u$$

$$+ \int_\Omega \langle \nabla u, \nabla(w-u)\rangle + f(w-u)$$

$$+ \int_\Omega \frac{1}{2}|\nabla(w-u)|^2 .$$

The middle term vanishes since $w - u = 0$ at $\partial\Omega$ (This term is just the 1st varn in direction $\varphi = w - u$) and the last term is strictly positive (since $w = u$ at $\partial\Omega$). Note that the last term is 2nd derv wrto t of value of the fnl at $u + t(w-u)$.

For (b): arg't is similar, leading to:

$$\text{value of fnl at } w = \text{value of fnl at } u + 0 + \int_\Omega \tfrac{1}{2}|\nabla(w-u)|^2$$

This time 3rd term can be 0, but only if $w - u$ is constant. That's right: the functional is minimized not only at u but also at $u + \text{const}$.

- For a rigorous treatment one must show that
  - var'l prin (a) achieves its min (esp: the min is not $-\infty$, and the bc is not lost)
  - var'l prin (b) achieves its min

if the data are consistent (esp: the min is not $-\infty$).

These are not esp difficult, but they lie beyond the scope of this class (see §4.5 of John, or the 1st lecture from my Calc of Var'n's class in Spring 2013).

Going further: let's discuss why var'l prins like (a) + (b) lead to successful numerical schemes, focusing for simplicity on

$$\Delta u = f \text{ in } \Omega, \quad u = 0 \text{ at } \partial\Omega.$$

Numerically, we would

(*) minimize $\int_\Omega \frac{1}{2}|\nabla u|^2 + fu$ over some finite-dim'l space S of functions on Ω with the given bc ($u = 0$ at $\partial\Omega$).

This would normally be done by seeking

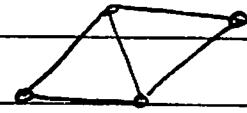
$$u \in S \text{ st } \int_\Omega \langle \nabla u, \nabla \varphi \rangle + f\varphi = 0 \text{ for all } \varphi \in S,$$

which amts to a finite-dim'l lin alg pblm.

Typical choices of S:

- span of 1st N basis fns from a well-chosen basis (eg eigenfns of Laplacian, if known, eg in a rectangle; or perhaps a wavelet basis, down to a specified level of fine-ness)

- piecewise linear fns on a specified triangulation (this is the simplest example of a finite element method). Note that a piecewise linear fn is entirely determined by its nodal values; also, there are no restns on the nodal values (a piecewise lin fn is automatically conts across edges)



First-order opt'y cond for (*) is this: at the optimal $u_S \in S$ we have

$$\int_\Omega \langle \nabla u_S, \nabla \varphi \rangle + f\varphi = 0 \text{ for all } v \in S.$$

How well does this work? <u>Most basic result</u>:

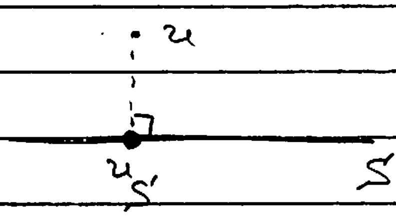
(**) $$\int_\Omega |\nabla(u-u_S)|^2 = \min_{v\in S} \int_\Omega |\nabla(u-v)|^2$$

ie the approx error (measured in the natural "energy" norm) is optimal — limited <u>only</u> by how well the soln of the pde can be approximated there (which depends on the smoothness of u, and the choice of S).

<u>Pf of (**)</u>: From 1st order opt'y condns of both pde + numerical scheme we get

$$\int_\Omega \langle \nabla(u_S-u), \nabla\varphi\rangle = 0 \quad \text{for all } \varphi\in S.$$

This means u_S = orthog projn of u onto S wrt inner product $(u,v) = \int_\Omega \langle \nabla u, \nabla v\rangle$.



Thus, from lin alg., u_S is the closest pt in S to u. <u>Done</u>.

[Here's a more elementary but actually equivalent version of the argument: for any $v \in S$ write $u - v = u - u_S + u_S - v$. Then

$$\int_\Omega \tfrac{1}{2}|\nabla(u-v)|^2 = \int_\Omega \tfrac{1}{2}|\nabla(u-u_S)|^2 + \tfrac{1}{2}|\nabla(u_S - v)|^2$$
$$+ \int_\Omega \langle \nabla(u-u_S), \nabla(u_S - v)\rangle$$

Last term $= 0$ since $u_S - v \in S$. So

$$\int_\Omega |\nabla(u-v)|^2 \geq \int_\Omega |\nabla(u-u_S)|^2$$

with equality only when $\int_\Omega |\nabla(u_S - v)|^2 = 0$ i.e. $u_S = v$.

For more on this topic see §8.5 of Strauss and §11.5 of Guenther + Lee (for short discussions) + 1st 50 pp of G. Strang + G. Fix, "An analysis of the finite element method" (for a readable tutorial with much more detail.)

One more piece of suggested reading: I assigned, as a HW pbm, the fact that if u is harmonic then so is the "Kelvin transform of u", $v(x) = |x|^{2-n} u(x/|x|^2)$. In Guenther + Lee §8.5 you'll see this applied to study the behavior of harmonic fns in exterior domains as $|x| \to \infty$.