PDE - Lecture 6, 10/8/2013

[Note: no class 10/15 (fall break); midterm 10/22]

Last lecture discussed some qualitative properties of harmonic fns (eg Mean Value Property) and a solution formula for $-\Delta u = f$ in $\mathbb{R}^n$ (via fundamental soln).

Today: more solution formulas (for bdry-value pbms, via Green's fns), and more qualitative properties (eg max principle).

Let's focus now on bvp with Dirichlet bc:

$$(*) \qquad \begin{aligned} -\Delta u &= f \quad \text{in } \Omega \\ u &= u_0 \quad \text{at } \partial\Omega \end{aligned}$$

We already know explicit soln formulas in special cases (eg Ω = a ball in $\mathbb{R}^2$ $\Rightarrow$ Fourier series of u_0 was used in HW 4; series can be summed by hand to get Poisson integral formula; something similar can also be done in $\mathbb{R}^n$, $n \geq 3$). But we seek something that applies more generally.

<u>Claim</u>: suppose $G(x,y)$ is defined for all $x,y \in \Omega$ $(x \neq y)$ and

$$-\Delta_y G(x,y) = \delta_x$$
$$G(x,y) = 0 \quad \text{for } y \in \partial\Omega$$

Then soln of (*) is

$$u(x) = \int_\Omega G(x,y) f(y)\, dy - \int_{\partial\Omega} u_0(y)\, \nabla_y G(x,y) \cdot \vec{n}$$

<u>Pf</u>: In general,

$$\int_\Omega v \Delta u - \int_\Omega u \Delta v = \int_{\partial\Omega} v \frac{\partial u}{\partial n} - \int_{\partial\Omega} u \frac{\partial v}{\partial n}.$$

Fixing x, take $v(y) = G(x,y)$; Then using (*),

$$\int_\Omega u \Delta v = -\int_\Omega u\, \delta_x = -u(x),$$

$$\int_\Omega v \Delta u = -\int_\Omega G(x,y) f(y)\, dy$$

$$\int_{\partial\Omega} u \frac{\partial v}{\partial n} = \int_{\partial\Omega} u_0 \nabla_y G(x,y) \cdot \vec{n}$$

$$\int_{\partial\Omega} v \frac{\partial u}{\partial n} = 0 \quad \text{since } v = 0 \text{ on } \partial\Omega.$$

G is called the "Green's fn" of Ω. (Note the analogy to – but also difference from – the Green's fn we studied earlier for the heat eqn.)

Does such a G exist? Yes, for any "reasonable" domain. We can "construct" it by taking

$$G(x,y) = \underbrace{\Phi(x-y)}_{\text{fund soln}} + \underbrace{\varphi^{(x)}(y)}_{\substack{\text{soln of } \Delta_y \varphi^{(x)} = 0 \text{ in } \Omega \\ \text{with bdry data} \\ \varphi^{(x)} = -\Phi(x-y),\ y \in \partial\Omega}}$$

For special cases, though, G can be made very explicit. Examples:

Half space: let $x = (x', x_n) \in \mathbb{R}^n$, and $\Omega = \{x_n > 0\}$. Then

$$G(x,y) = \Phi(y-x) - \Phi(y-\tilde{x})$$

where $\tilde{x}$ = "reflection of x" $= (x', -x_n)$. Defining properties of G are clear by inspection. [Note resemblance to what we did for $u_t - \Delta u = 0$. When bdry cond $u_0 = 0$, soln of $-\Delta u = f$ in half space is obtained by taking $\tilde{f}$ = odd extnsn of f, then solving $-\Delta\tilde{u} = \tilde{f}$ via fund soln; since $\tilde{u}$ is odd, its restrn to halfspace is the desired soln.]

<u>Ball</u>: same idea, but "reflection" is replaced by inversion; for $\Omega = B(0,1)$,

$$G(x,y) = \Phi(y-x) - \Phi(|x|(y-\tilde{x}))$$

where $\tilde{x} = x/|x|^2$. (Crucial fact: we have $|y-x| = |x||y-\tilde{x}|$ when $x \in B(0,1)$ and $y \in \partial B(0,1)$.)

When $f=0$ (so we're solving $\Delta u = 0$, $u = u_0$ at bdry) assoc repn formulas are "Poisson's formula" (for a halfspace or ball respectively).

Evans' Section 2.2.4 is a good place to find more detail on this topic (of course there are versions in almost every book).

Important fact abt Greens' fn: it's <u>symmetric</u>, ie $G(x,y) = G(y,x)$. Here is a "formal pf": recall Green's formula

$$\int_\Omega u\Delta v - v\Delta u = \int_{\partial\Omega} u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n}.$$

Take $u(x) = G(P,x)$ + $v(x) = G(Q,x)$, with $P \neq Q$. Then $-\Delta u = \delta_{x=P}$, $-\Delta v = \delta_{x=Q}$, and both vanish at $\partial\Omega$. So we get

$$-\int_\Omega u\,\delta_{x=Q} + \int_\Omega v\,\delta_{x=P} = 0$$

ie

$$G(P,Q) = G(Q,P).$$

(This proof is easily made honest, by the same argument we used to see that Φ was the fundamental solution.)

Similar technique can be used to solve Neumann bv problems. Just use $N(x,y)$ (the "Neumann function"), defined by

$$-\Delta_y N(x,y) = \delta_x \quad \text{for } y \in \Omega$$

$$\frac{\partial N}{\partial \nu_y} = \text{const} \qquad \text{for } y \in \partial\Omega$$

(The value of the constant in the bc is determined by condn that $\int_{\partial\Omega} \partial N/\partial \nu_y = \int_\Omega \Delta_y N = -1$.)

Recall that the bvp

$$-\Delta u = f \quad \text{in } \Omega$$
$$\partial u/\partial n = g \quad \text{at } \partial\Omega$$

has a soln only if data are consistent:

$$\int_{\partial\Omega} g + \int_{\Omega} f = 0,$$

and that (for consistent data) soln is unique only up to an additive constant. We can choose the const so that $\int_{\partial\Omega} u = 0$

<u>Claim</u>: For the soln normalized this way,

$$u(x) = \int_{\Omega} N(x,y) f(y) + \int_{\partial\Omega} N(x,y) g(y).$$

<u>Pf</u>:
$$\int_{\Omega} u \Delta N - N \Delta u = \int_{\partial\Omega} u \frac{\partial N}{\partial \nu} - N \frac{\partial u}{\partial \nu}$$

$$\Rightarrow \quad -u(x) + \int_{\Omega} Nf = 0 - \int_{\partial\Omega} Ng$$

As for the Green's function, N is symmetric (same pf as for G) and it is easy to write N in terms of the fund soln plus a correction obtained by solving a (Neumann) bvp in Ω.

Enough about representation formulas. Let's discuss qualitative properties. Key pts are

A) there's a max principle (with many parallels to what we did previously for the heat eqn)

B) harmonic fns are smooth (though bdry value need not be smooth!); soln of $\Delta u = f$ is as smooth as permitted by f

About (A): "wk max prin" says, for bdd domains,

$$\Delta u = 0 \text{ in } \Omega \Rightarrow u \text{ achieves its max + min values at } \partial\Omega.$$

A proof can be done along lines we used for heat eqn (when $\Delta u < 0$ or $\Delta u > 0$ argt is exactly same; reduce general case to this one by considering $u_\varepsilon = u \pm \varepsilon |x|^2$).

But for $\Delta u = 0$ we can use MVP to prove a stronger stmt – the "strong max prin": if max or min of u is achieved at an interior pt then u must be constant (if Ω is connected)

Pf: suppose $\max_{x \in \bar\Omega} u(x) = M$, and consider the set

$$S = \{x \in \Omega \, ; \, u(x) = M\}.$$

It's closed, since u is cont's (recall: harmonic fns are smooth). But it's also open, by MVP:

if $u \le M$ in $B_r(x)$ + $u(x) = \fint_{B_r} u = M$

then $u \equiv M$ throughout B_r.

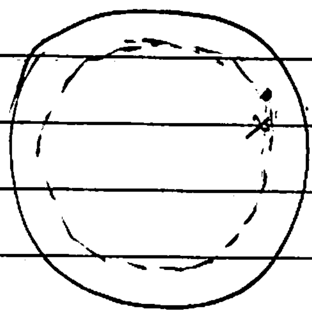
So if Ω is connected, S is either empty or else all Ω. (If Ω is not connected, we still get $u < M$ or else $u \equiv M$ on each connected component.)

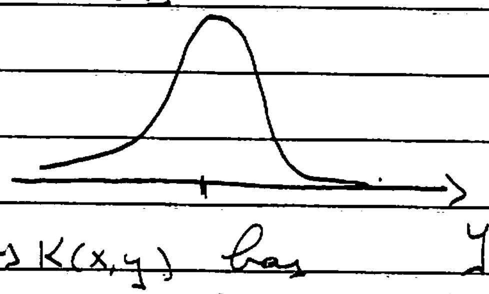
As in parabolic setting, max prin has lots of uses (see HW6).

About (B): intuition is that for $\Delta u = 0$ in Ω, $u = g$ at $\partial\Omega$, soln inside is a smoothed-out version of bdry data (just as soln of $u_t - \Delta u = 0$ for $t > 0$ is a smoothed-out version of initial data). Can be made very explicit for a ball or halfspace, using Poisson integral formula (obtained from the Green's fn repn). Schematic in $B = B_1(0)$:

$\Delta u = 0$ in B, $u = g$ at ∂B.

$$\Longrightarrow \quad u(x,y) = \int_{\partial B} K(x,y)\, g(y)\, dy$$





$y \to K(x,y)$ has its peak at x; peak is sharper as x approaches ∂B (it's not Gaussian, but the net effect is similar)

Other ways of capturing this regularizing effect:

- gradient bounds: when we proved Liouville's Thm in Lecture 5, the first step was a gradient bound, proved using the MVP:

$$\Delta u = 0 \text{ in } B_r(x_0) \Rightarrow |\nabla u(x_0)| \le \frac{\text{const}}{r} \max_{\partial B_r(x_0)} |u|$$

So: if u is merely bdd at $\partial\Omega$, it is locally Lipschitz in the interior of Ω (but the local Lipschitz const $\to \infty$ like $1/(\text{dist to bdry})$ as one approaches the bdry). Since $\partial u/\partial x_i$ also solves Laplace's eqn, this est can be iterated (you find to do something similar in HW5), eg

$$\Delta u = 0 \text{ in } B_r(x_0) \Rightarrow |\nabla^2 u(x_0)| \le \frac{\text{const}}{r^2} \max_{\partial B_r(x_0)} |u|.$$

- Harnack's inequality (just the simplest version): if u is harmonic and nonnegative in B_r, then

$$\max_{B_{r/2}} u \le C \min_{B_{r/2}} u.$$

ie the values of u are all comparable in the (concentric) half-ball. (The const C is indep of r.)

Pf (taken from Han, §4.2). When u is nonneg our gradient bound becomes

$$\frac{\partial u}{\partial x_i}(x_0) = \frac{c}{\rho} \fint_{\partial B_\rho(x_0)} u \cdot \nu_i \le \frac{c}{\rho} \fint_{\partial B_\rho(x_0)} |u|.$$

(see pg 4 of Lecture 5)

$$= \frac{c}{\rho} \fint_{\partial B_\rho(x_0)} u = \frac{c}{\rho} u(x_0)$$

(using $u \ge 0$ and the MVP). This is true at any pt x_0 st $B_\rho(x_0) \subset B_r$. Taking $\rho = r/2$, we

$$|\nabla u(x)| \le \frac{c_1}{r} u(x) \quad \text{for all } x \in B_{r/2}$$

(The constant c_1 is indep of r, either by inspecting the proof, or by observing that the assertion

$$\Delta u = 0 + u \ge 0 \text{ in } B_r \Rightarrow |\nabla u| \le \frac{c_1}{r} u \text{ in } B_{r/2}$$

is scale-invariant.)

We may assume $u > 0$ (otherwise consider $u + \varepsilon$ instead of u, passing to limit $\varepsilon \to 0$ at end of arg t).
Then preceding result says

$$|\nabla \log u(x)| \le \frac{c_1}{r} \quad \text{in } B_{r/2}$$

For any $x, y \in B_{r/2}$ we get

$$\log \frac{u(x)}{u(y)} = \int_0^1 \frac{d}{dt} \log u(tx + (1-t)y)\, dt$$

$$= (x-y) \cdot \int_0^1 \nabla \log u(tx + (1-t)y)\, dt$$

$$\le |x-y| \int_0^1 |\nabla \log u(tx + (1-t)y)|\, dt$$

$$\le |x-y| \frac{c_1}{r} \le c_1$$

so $$\frac{u(x)}{u(y)} \le e^{c_1}.$$

That's what we wanted.

Remaining tasks:

a) discuss how solns to $\Delta u = f$ (with suitable bc) can be found numerically (and, along the way, an indication why they exist)

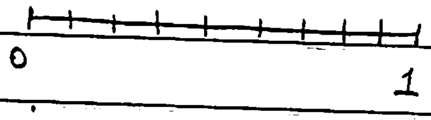
b) some discn of pblms that are similar but nonlinear.

There are several approaches to (a), including

- boundary integral eqns (thoroughly discussed by Guenther + Lee)
- Perron's method (nicely discussed by F. John)
- variational method

We'll discuss only the variational method, which has the advantage of extending nicely to many nonlinear problems. That will be in Lecture 7.

But first (since it's easy) let's look briefly at finite difference schemes for Laplace's eqn (for a simple discn see W. Strauss's book).

1D:  $x_j = \frac{j}{N}$ $\quad x_0 = 0, \ x_N = 1$

Finite diff version of Dir pbm $u_{xx} = f$ on $[0,1]$, $u=0$ at $x=0$ + $x=1$:

$$\frac{u_{j+1} + u_{j-1} - 2u_j}{(\Delta x)^2} = f_j \qquad j = 1, \dots, N-1$$

with convention $u_0 = 0$, $u_N = 1$. This can be expressed as a matrix eqn $K\vec{u} = \vec{f}$ (w/ $\Delta x = \frac{1}{N}$):

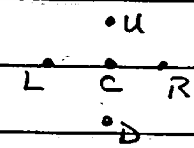
$$\frac{1}{(\Delta x)^2}\begin{bmatrix} -2 & 1 & & & 0 \\ 1 & -2 & 1 & & \\ & 1 & -2 & 1 & \\ & & \ddots & & \\ 0 & & 1 & -2 & 1 \\ & & & 1 & -2 \end{bmatrix} \begin{bmatrix} u_1 \\ \vdots \\ \vdots \\ u_{N-1} \end{bmatrix} = \begin{bmatrix} f_1 \\ \vdots \\ \vdots \\ f_{N-1} \end{bmatrix}$$

Matrix is tridiagonal (and very well understood). Not surprisingly, it's invertible; in fact same algebra =>

$$\langle K\vec{u}, \vec{u} \rangle = -\sum_{j=0}^{N-1} \left| \frac{u_j - u_{j+1}}{\Delta x} \right|^2$$

(with convention $u_0 = 0$, $u_N = 0$). This is the discrete analogue of $\int_0^1 u \Delta u = -\int_0^1 |\nabla u|^2$ if $u = 0$ at $x = 0, 1$.

2D: Finite differences work fine in a <u>rectangle</u>. But now discrete Laplacian involves 5 pt stencil



$$\Delta u(C) \approx \frac{u_R+u_L+u_U+u_D-4u_C}{(\Delta x)^2}$$

and matrix is no longer tridiagonal. (There is, however, still a discrete analogue of integn by parts, as in 1D).

Bottom line: solving Laplace's eqn with a Dir bc amounts to inverting a linear system (in fact, one with rather special structure).