PDE - Lecture 5, 10/1/2013

Recall that for heat eqn $u_t = \Delta u$ our discn included

- qualitative properties (uniqueness, max principle, smoothness, decay)
- "explicit" solution formulas (by separation of variables, using the fundamental soln, etc)
- some simple numerical approx schemes

As we turn now to Laplace-type eqns such as

- $\Delta u = f$ in all $\mathbb{R}^n$, with a condn at ∞ to assure uniqueness
- $\Delta u = f$ in a bounded domain Ω, with $u = \varphi$ at $\partial\Omega$
- $\Delta u = f$ in a bounded domain Ω, with $\partial u/\partial\nu = \varphi$ at $\partial\Omega$

our goal is to assemble an analogous understanding.

A comment on readings: F John's is good to read on this topic, but I find his discn very compressed. Kevorkian is OK but as usual he is (overly) concerned with explicit solution formulas. Guenther + Lee emphasize bdry integral eqns, a valuable topic but one that doesn't fit in the few weeks we have for this topic. To my taste, the best concise discn of the basics is in Evans (Chapter 2.2, occupies abt 20 pp)

First topic: Mean value property ("MVP"); we start with this since
 a) it's elegant + elementary
 b) it permits an easy pf of uniqueness for $\Delta u = f$ in $\mathbb{R}^n$ (with suitable conds at ∞).

Mean value property: if u is C^2 and $\Delta u = 0$ then

$$u(x) = \fint_{\partial B(x,r)} u \, dA = \fint_{B(x,r)} u \, d\mathrm{vol}$$

where the slashed integrals denote averages (eg in $\mathbb{R}^2$: $\fint_{\partial B(0,r)} u = \frac{1}{2\pi} \int_0^{2\pi} u(re^{i\theta}) \, d\theta$, and $\displaystyle\fint_{B(0,r)} u = \frac{1}{\pi r^2}\int_0^r\int_0^{2\pi} u(\rho e^{i\theta})\,\rho\,d\rho\,d\theta.$)

Pf: Let $\displaystyle\varphi(r) = \fint_{\partial B(x,r)} u = \frac{1}{|\partial B(0,1)|}\int_{\partial B(0,1)} u(x+ry)\,d\text{area}_y$

Evidently

$$\varphi'(r) = \fint_{\partial B(x,r)} \frac{\partial u}{\partial \nu}$$

But $\displaystyle\int_{\partial B(x,r)} \frac{\partial u}{\partial\nu} = \int_{B(x,r)} \Delta u = 0.$ So φ is indep of r. But clearly (since u is conts)

$$\varphi(r) \to u(x) \quad \text{as } r \to 0.$$

So $\varphi(r) = u(x)$ for all x, proving the 1st part of the MVP

$$\fint_{\partial B(x,r)} u = u(x) \quad \text{for all } r.$$

The second assertion follows easily by integn wrt r ("method of shells")

$$\int_{B(x,r)} u\,d\text{vol} = \int_0^r \left(\int_{\partial B(x,\rho)} u\,d\text{Area}\right) d\rho.$$

$$= u(x) \cdot \int_0^r \text{Area of } \partial B(x,\rho) \, d\rho$$

$$= u(x) \cdot |B(x,r)| .$$

MVP has many imptnt consequences – we'll return to some of them later – but the one we want now is this:

<u>Liouville's Thm for harmonic fns in $\mathbb{R}^n$</u>: if $\Delta u = 0$ in $\mathbb{R}^n$ and u is unif bdd then $u \equiv$ const.

<u>Pf</u>: Observe (assuming u is C^3) that

$$\Delta u = 0 \implies \Delta \frac{\partial u}{\partial x_i} = 0 .$$

Now

$$\frac{\partial u}{\partial x_i}(x_0) = ⨍_{B(x_0,r)} \partial u/\partial x_i \qquad \text{by MVP}$$

$$= \frac{c}{r^n} \int_{\partial B(x_0,r)} u \cdot \nu_i$$

using that $\quad \partial u/\partial x_i = \langle \nabla u, \nabla \varphi \rangle \quad$ with $\varphi = (x - x_0)_i$
$= \operatorname{div}(u \nabla \varphi)$

so that $\displaystyle\int_{B(x_0,r)} \partial u/\partial x_i = \int_{\partial B(x_0,r)} u \frac{\partial\varphi}{\partial\nu}$

and that on $\partial B(x_0,r)$, $\varphi = |x-x_0| \dfrac{(x-x_0)_i}{|x-x_0|} \Rightarrow \dfrac{\partial\varphi}{\partial\nu} = \nu_i$.

If u is unif bdd, we thus get

$$\left|\frac{\partial u}{\partial x_i}(x_0)\right| \le \frac{\text{const}}{r} \,|\max u|$$

And as $r \to \infty$ we see $\nabla u = 0$.

Rmk: in fact we have proved more: the argt shows that if

$$\frac{|u(x)|}{|x|} \to 0 \quad \text{as } |x| \to \infty$$

then u is constant. Thus: a nonconst harmonic fn on all $\mathbb{R}^n$ must grow at least linearly at ∞. (And indeed there are examples: linear fns are harmonic!). This will be importnt shortly.

Hypoth above was that u was C^3. Actually, $\Delta u = 0$ in any reasonable sense $\Rightarrow u$ is C^∞. But it's not convenient to prove that now. (For an easy pf via MVP see Evans.)

Let's turn now to a repn formula for solns of

$$\Delta u = f \quad \text{in all } \mathbb{R}^n$$

subject to condn as $|x| \to \infty$ that assures uniqueness (up to a constant), for example

$$u \to 0 \quad \text{as} \quad |x| \to \infty$$ (reasonable if $n \geq 3$ and f has cpt spt)

or

$$\frac{|u(x)|}{|x|} \to 0 \quad \text{as } |x| \to \infty$$ (more natural if $n = 2$, as we'll see soon).

The main pt: in $\mathbb{R}^n$, $n \geq 2$,

$$(*) \qquad u(x) = \int_{\mathbb{R}^n} \Phi(x-y)\, f(y)\, dy$$

solves $-\Delta u = f$ when

$$(**) \qquad \Phi(z) = \begin{cases} \frac{-1}{2\pi} \log |z| & \text{in } \mathbb{R}^2 \\ C_n |z|^{2-n} & \text{in } \mathbb{R}^n,\ n \geq 3 \end{cases}$$

with the right choice of c_n, namely

$$c_n = \frac{1}{n(n-2)\alpha_n} \qquad \alpha_n = \text{vol of unit ball in } \mathbb{R}^n.$$

We'll prove this assuming f is C^2. (but it's true in much greater generality). Φ is called the "fundamental solution".

What's special about the Φ defined by (**)? It's radial, smooth away from $z=0$, and $\Delta_z \Phi = 0$ away from $z=0$. Easy to check the last stmt: $\Delta \Phi = \Phi_{rr} + \frac{n-1}{r}\Phi_r$ for radial fns, so

$$\Delta \Phi = 0 \implies \Phi_{rr} = -\frac{n-1}{r}\Phi_r$$

$$\implies (\log \Phi_r)_r = \frac{\Phi_{rr}}{\Phi_r} = \frac{1-n}{r}$$

$$\implies \log \Phi_r = c_1 + (1-n)\log r$$

$$\implies \Phi_r = c_2 r^{1-n}$$

$$\implies \Phi = \begin{cases} a + b\log r & n=2 \\ a + \dfrac{b}{r^{n-2}} & n \geq 3 \end{cases}$$

The additive constant isn't interesting; the multiplicative one ("b") is a normalization.

OK, let's think abt $u(x) = \int \Phi(x-y) f(y)\, dy$.
First thought: differentiate under the integral.
Is this ok?

well: $$\frac{\partial u}{\partial x_i} = \int \frac{\partial}{\partial x_i} \Phi(x-y)\, f(y)\, dy$$

since $\nabla \Phi$ is abs integrable (eg on $\mathbb{R}^2$: $\frac{1}{r}$ is integrable near 0).

but: calculating $\frac{\partial^2 u}{\partial x_i \partial x_j}$ this way is wrong!

In fact $\nabla\nabla\Phi$ is not integrable at 0, so there's no reason it should be correct. (There is a formula of the form

$$\frac{\partial^2 u}{\partial x_i \partial x_j}(x) = \int_{PV} \frac{\partial^2}{\partial x_i \partial x_j} \Phi(x-y)\, f(y)\, dy + c_{ij} f(x)$$

where the integral is a "Cauchy principal value", but this would take us too far afield.)

So let's be more careful:

<u>step 1</u> If f is C^2 and $u(x) = \int \Phi(x-y) f(y)\, dy$ then u is C^2.

Pf: let $z = x - y$ so $y = x - z$. Then

$$u(x) = \int \Phi(z) f(x-z)\, dz$$

Now differentiate under the integral.

step 2 $$\Delta u(x) = \int_{|z|<\varepsilon} \Phi(z)\, \Delta_x f(x-z)\, dz + \int_{|z|>\varepsilon} \Phi(z)\, \Delta_x f(x-z)\, dz.$$

1st term $\to 0$ as $\varepsilon \to 0$ (using character of Φ near $z=0$, and hypothesis that f is C^2).

2nd term is the interesting one. Integrate by pts, after observing that $\Delta_x f(x-z) = \Delta_z f(x-z)$:

$$\int_{|z|>\varepsilon} \Phi(z)\, \Delta_z f(x-z)\, dz = \int_{|z|>\varepsilon} -\langle \nabla \Phi, \nabla_z f(x-z) \rangle + \int_{|z|=\varepsilon} \Phi \frac{\partial f}{\partial \nu}$$

and agn the bdry term $\to 0$ as $\varepsilon \to 0$.

So: the really interesting term is

$$\int_{|z|>\varepsilon} -\langle \nabla\Phi, \nabla_z f(x-z) \rangle$$

$$= \int_{|z|=\varepsilon} \frac{\partial \Phi}{\partial \nu} f(x-z)$$

using that $\Delta\Phi = 0$ away from $z=0$. Now,

$$\frac{\partial\Phi}{\partial\nu} = \frac{\partial\Phi}{\partial r} = \frac{\text{const}}{\varepsilon^{n-1}} \qquad (\text{constant on } |z|=\varepsilon)$$

and $f(x-y) \approx f(x)$ by cont'y of f. So this "really interesting term" $\xrightarrow[\varepsilon\to 0]{}$ a constant times $f(x)$.

The const in the fund soln is chosen so we get exactly $f(x)$.

Common notation: $\Delta\Phi = \delta_0$. RHS is a "delta fn" ie a pt mass at 0. Explain this: recall from step 1 that

$$\Delta u(x) = \int \Phi(z)\, \Delta_z f(x-z)$$

$$= \int \langle -\nabla_z \Phi, \nabla_z f(x-z) \rangle \, dz \qquad \underline{\text{honestly}}$$

$$= \int \Delta_z \Phi \cdot f(x-z)\, dz \qquad \underline{\text{formally}}$$

In fact $\Delta u(x) = f(x)$ (we proved this). So the "formal" soln is true if we interpret $\Delta_z \Phi = \delta_0$.

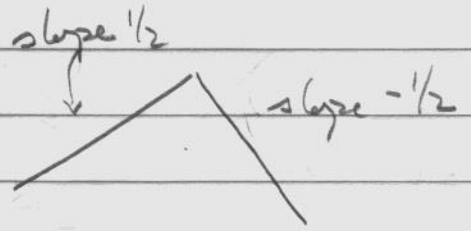
<u>Alternative viewpt</u>: one can regularize the fund soln, eg

$$\Phi_\varepsilon(z) = \frac{c_n}{(|z|^2+\varepsilon^2)^{\frac{1-n}{2}}} \qquad \text{if } n \geq 3 .$$

Then $\Delta \Phi_\varepsilon$ makes perfect sense for $\varepsilon > 0$ and we can study it in the limit $\varepsilon \to 0$. See eg Folland.

Another attempt at intuition: in 1D,

$$-\Delta \Phi = -\Phi_{zz} = \delta_0 \quad \text{when}$$



since Φ_z is then piecewise constant with a jump of -1 at $z=0$.

One can make sense of this by proving

$$-\int_{-\infty}^{+\infty} \Phi(z)\, f''(z)\, dz = f(0) \text{ for all nice,}$$

cptly sptd f

(use integn by parts on $(-\infty,0)$ + $(0,\infty)$ separately), or by smoothing Φ then taking a limit.

The situation for our multidim'l Φ is similar, except that $\Phi(0)$ is undefined, so the integr by parts is best done on $|z| > \varepsilon$.

<u>Yet another</u>: remember that $\displaystyle\int_{\partial B_r(0)} \frac{\partial \Phi}{\partial \nu} = \int_{B_r(0)} \Delta \Phi$; since $-\Delta \Phi = \delta_0$ we expect

$$\int_{\partial B_r(0)} \frac{\partial \Phi}{\partial r} = -1$$

This is a good way to pin down the normalization constant c_n: LHS is (if $n \geq 3$)

$$c_n \left(\frac{d}{dr} r^{2-n} \right) \cdot r^{n-1} \cdot \begin{pmatrix} \text{area of unit} \\ n-1 \text{ sphere in } \mathbb{R}^n \end{pmatrix}$$

so

$$c_n \cdot (2-n) \cdot n \alpha_n = -1$$

(similar calcn can be done in $\mathbb{R}^2$).

Our soln formula is useful for more than just "knowing the soln". One consequence is <u>regularity</u>:

if $\Delta u = 0$ in a nbhd of x_0,
then u is C^∞ near x_0.

Pf: consider $\tilde{u}(x) = u \cdot \varphi(x)$, where

$\varphi \equiv 1$ near x_0
$\varphi \equiv 0$ where $\Delta u \neq 0$ (or where u is not defined).

Then $\Delta \tilde{u} = \underbrace{2\nabla u \cdot \nabla\varphi + u\Delta\varphi}_{\text{call this } f}$ in all $\mathbb{R}^n$

and $\tilde{u} = 0$ near ∞. So (using our uniqueness result)

$$\tilde{u}(x) = \int_{\mathbb{R}^n} \Phi(x-y)\, f(y)\, dy$$

and $f \equiv 0$ near $x_0 \Rightarrow$ we can differentiate under the integral ($x \to \Phi(x-y)$ is a smooth fn of x in a nbhd of x_0, if y is restricted to spt of f).

For heat eqn we used
- fund soln in $\mathbb{R}^n$
- sep of vars in bdd domains
- Green's fn, as a way to think abt both the sep of vars soln and other cases (eg halfspace)

For $\Delta u = f$ (or $\Delta u = 0$ with bdry data) corresp picture is

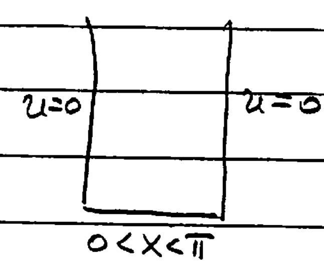
fund soln, for $\Delta u = f$ in $\mathbb{R}^n$

— sep of vars works for some special domains (eg HW4 explored the case of a disk in $\mathbb{R}^2$)

Green's fns again provide a more general perspective.

Before turning to Green's fns, let's examine a few settings where sep of vars works well

<u>example 1</u>: <u>a half-strip in $\mathbb{R}^2$</u>, with $u=0$ at sides as shown



$\Rightarrow$ use $\sin kx \, e^{-ky}$ $\quad k=1,2,\dots$

(Effectively: represent $x \to u(x,y)$ as a Fourier sine series.

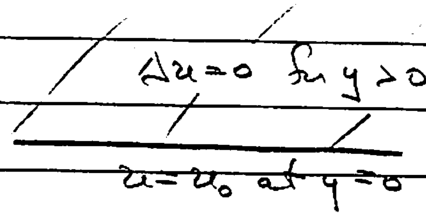
$$u(x,y) = \sum a_k(y) \sin kx$$

and solve an ode for $a_k(y)$ to see that

$$a_k(y) = \text{const}\, e^{ky} + \text{const}\, e^{-ky} \; ;$$

a growth cond^n at $y \to \infty$ would rule out the growing exponentials)

<u>example 2</u>: <u>a halfspace</u> eg



Almost the same as ex 1, except now there's no restn on k, and you must use Fourier transform not Fourier series in x

$$u(x,y) = C \int_{-\infty}^{+\infty} \hat{u}_0(\xi)\, e^{-|\xi| y}\, e^{i \xi \cdot x}$$

<u>example 3</u>: periodic bdry conds, eg
$\Delta u = f$ where $u + f$ are periodic with period 2π in each variable (note consistency cond: $\int_{\substack{\text{period}\\ \text{cell}}} f = 0$ since $\int_{\text{cell}} \Delta u = \int_{\partial(\text{cell})} \partial u / \partial \nu = 0$ when u is periodic).

Using complex notation:

$$f \text{ periodic} \iff f(x) = \sum_{k \in \mathbb{Z}^2} \hat{f}(k)\, e^{i k \cdot x}$$

$$\Delta u = f \iff -|k|^2 \hat{u}(k) = \hat{f}(k). \qquad k \neq 0$$

(note: consistency $\iff$ $\hat{f}(0) = 0$; also soln is only unique up to a const, and indeed $\hat{u}(0)$ is arbitrary).

Periodic case makes it easy to see that

f has s derivs in L^2 $\Rightarrow$ $\frac{\partial^2 u}{\partial x_i \partial x_j}$ has s derivs in L^2 for each i, j

$\Rightarrow$ u has s+2 derivs in L^2

(Sketch: use that for periodic fns with period 2π

$$\int_{cell} |f|^2 \, dx = c \sum_{k \in \mathbb{Z}^n} |\hat{f}(k)|^2 \qquad .)$$

Rmk: for heat eqn, we sketched how a formula for the fund soln can be deduced from a repn of soln of $u_t = u_{xx}$ via Fourier transform. Similarly, fund soln for $\Delta u = f$ can also be found using Fourier transform.

Coming next lecture:
- Green's fns (a more systematic approach to bdry value pbms)
- qualitative properties (eg max principle)