PDE – Lecture 4, 9/24/2013

We turn now to Laplace's eqn, + related problems. Some examples of how such pbms arise

(1) **heat transfer**: we discussed how

$$u_t - \Delta u = f \quad \text{in } \Omega, \quad t > 0$$
$$u = \varphi \quad \text{at } \partial\Omega$$

arises as a simple model for the evolution of the temperature u in a body; here f comes from sources or sinks of heat in Ω, + the bc assumes temp is fixed at $\partial\Omega$. As $t \to \infty$, we expect decay to

$$-\Delta u = f \quad \text{in } \Omega$$
$$u = \varphi \quad \text{at } \partial\Omega$$

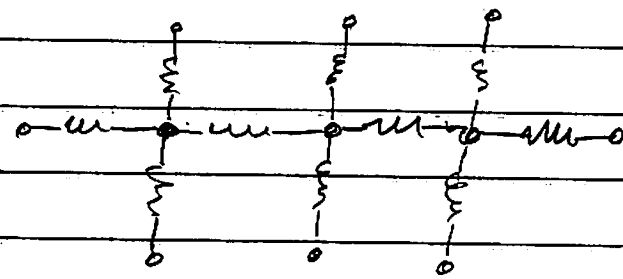
(if $\varphi + f$ are indep of time).

Same story applies in all $\mathbb{R}^n$, eg with $u(x,t) \to 0$ as $|x| \to \infty$. Then we expect decay as $t \to \infty$ to soln of

$$-\Delta u = f \quad \text{in } \mathbb{R}^n.$$

(cond at ∞ plays role of a bc)

(2) <u>continuum limit of current flow in a resistor network:</u> work in 2D for simplicity



spatial mesh h;

u_{ij} = voltage at i,j^{th} node
(spatial pt ih, jh)

current flow from i,j to $i+1,j$ is
proportional to $u_{i+1,j} - u_{i,j}$ (Ohm's law)

similar convention for vertical links.

total current balance in/out of node i,j:

$$u_{i+1,j} + u_{i-1,j} + u_{i,j+1} + u_{i,j-1} - 4u_{i,j} = 0$$

<u>Finite difference approxn of</u> $u_{xx} + u_{yy} = 0$

Note: if there are current sources at nodes, then (assuming appropriate scaling) we get by the same argt a finite difference approxn of $\Delta u = f$.

(3) probability - steady state distn.

We saw in HW 1 that a biased random walk provides finite-difference analogue of

$$u_t = u_{xx} - (\alpha(x)\, u)_x = 0$$

As $t \to \infty$, we may expect convergence to

$$u_{xx} - (\alpha(x) u)_x = 0,$$

representing the steady-state probability distn.

Warning: if $\alpha = 0$ and the initial data decay to 0 as $x \to \infty$ then $u \to 0$ as $x \to 0$. But in other cases there can be a nonzero limit. For example if bias always pushes walker toward 0, say

$$\alpha(x) = -x$$

then $u = Ce^{-\frac{1}{2}x^2}$ solves $u_{xx} + (xu)_x = 0$

(4) <u>probability</u> - expected arrival at bdry

In 1D as usual, moving left or right with prob 1/2 (unbiased) + time step Δt s.t. $(\Delta x)^2/2\Delta t = 1$

$u(j\Delta x)$ = expected time when walker reaches bdry, starting from $j\Delta x$ at time 0

$$u(j\Delta x) = \underbrace{\tfrac{1}{2}u((j-1)\Delta x) + \tfrac{1}{2}u((j+1)\Delta x)}_{\substack{\text{expected arrival time} \\ \text{starting from next step}}} + \Delta t$$

$$\Rightarrow \quad u_{j-1} + u_{j+1} - 2u_j = -2\Delta t = -(\Delta x)^2$$

Discretization of $u_{xx} = -1$. Bdry cond is $u = 0$ at $\partial\Omega$.

In 2D, if walker goes to each of 4 nearest neighbors with equal prob, get similarly a discretization of

$$u_{xx} + u_{yy} = -1 \qquad \text{if } \frac{4\Delta t}{(\Delta x)^2} = 1$$
$$(u = 0 \text{ at bdry})$$

(5) incompressible, irrotational flow (see eg Kevorkian, opening of Chap 2)

$\vec{q}$ = fluid velocity (a vector field)

$\operatorname{div} \vec{q} = 0 \iff$ incompressibility
$\operatorname{curl} \vec{q} = 0 \iff$ flow is "irrotational"

The latter implies $\vec{q} = \nabla \varphi$. Then $\operatorname{div} \vec{q} = 0 \iff \Delta \varphi = 0$.

(6) complex variables: if $f(z)$ is analytic then its real + imaginary parts (viewed as fns of x + y, where $z = x+iy$) are both harmonic.

(7) variational problems: if u_* achieves

$$\min_{u = \varphi \text{ at } \partial\Omega} \int_\Omega W(\nabla u)$$

Then for any v st $v|_{\partial\Omega} = 0$ we have

$$t \to \int_\Omega W(\nabla u_* + t \nabla v) \quad \text{has a min at } t = 0$$

so, by 1st deriv test, $\displaystyle\int_\Omega \operatorname{div}\left[\frac{\partial W}{\partial \nabla u}(\nabla u_*)\right] \cdot v \, dx = 0.$

If $\operatorname{div}\left[\frac{\partial W}{\partial \nabla u}(\nabla u_*)\right]$ were nonzero, there would be a function v that makes the integral nonzero. So we get the nec condn for optimality:

$$\operatorname{div}\left[\frac{\partial W}{\partial \nabla u}(\nabla u_*)\right] = 0.$$

Special cases:

a) solving $\Delta u = 0$ in Ω, $u = \varphi$ at $\partial\Omega$ $\iff$ $\min\limits_{\substack{u=\varphi \\ \text{at } \partial\Omega}} \int_\Omega |\nabla u|^2$.

b) solving $\min\limits_{\substack{u=\varphi \\ \text{at } \partial\Omega}} \int_\Omega (1+|\nabla u|^2)^{1/2}$ (area of graph of u).

$$\iff \operatorname{div}\left[\frac{\nabla u}{(1+|\nabla u|^2)^{1/2}}\right] = 0$$

This eqn is nonlinear, however if we expect ∇u to be small we can take the

small-slope approxn $(1+|\nabla u|^2)^{-1/2} \approx 1 - \frac{1}{2}|\nabla u|^2$.

and we see that the leading-order ("linearized") eqn is $\Delta u = 0$.

Let's start with some easy observations abt const-coefft Laplace-type eqns in a bounded domain.

(1) Eqn $\Delta u = f$ in Ω.

$u = \varphi$ at $\partial\Omega$

can have at most one soln. (In fact, there <u>is</u> a soln; we'll return to this in a couple of weeks).

<u>Pf</u>: subtracting 2 solns, sufft to show $f=0$ and $\varphi=0 \Rightarrow u=0$. Using "energy method": mult eqn by u to see

$$\int_\Omega u\Delta u = 0 \quad \Longrightarrow \quad \int_\Omega |\nabla u|^2 = 0.$$

$$\Longrightarrow \quad u = \text{const}$$

Now bdry condn $\Rightarrow u = 0$.

[Pf can also be done by max prin; we'll discuss max prin later]

(2) Eqn $\Delta u = f$ in Ω

$\frac{\partial u}{\partial n} = g$ at $\partial\Omega$

can have a soln only if $g + f$ satisfy the consistency condn

$$\int_{\partial\Omega} g \, dA = \int_{\Omega} f \, dx.$$

For consistent data, soln is unique up to a constant. (In fact it exists; agn, we'll discuss this later.)

Pf of consistency cond: $\int_{\Omega} \Delta u = \int_{\partial\Omega} \frac{\partial u}{\partial n}$ for any fn u, by Gauss' Thm.

Pf of uniqueness up to const: exactly as before, by "energy method".

What is happening? Recall from linear algebra that if A is a symmetric $n \times n$ matrix, then $Az = b$ has a soln iff $b \perp \ker A$.

Also recall that Δ is self-adj't for L^2 inner product, with either the bc $u = 0$ at $\partial\Omega$ or the bc $\partial u/\partial n = 0$ at $\partial\Omega$. We have a consistency condn for the Neumann pbm because there's a nontrivial kernel (the constant fns) solving $\Delta u = 0$ when the bc is $\partial u/\partial n = 0$.