PDE – Lecture 3, 9/17/2013

We turn now to the heat eqn in $\mathbb{R}^n$

$$u_t - \Delta u = 0 \quad \text{for } x \in \mathbb{R}^n,\ t > 0$$
$$u = u_0(x) \quad \text{at } t = 0$$

(and, a little later, related problems for example the heat eqn in a half-space).

What carries over from case of a bdd domain, + **what doesn't**?

- existence can no longer be done using L^2 eigenfunctions of Δ; instead we'll get an excellent "solution formula" using the "fundamental solution"

- a finite difference repn is possible, but no longer very practical (space is infinite!)

- implicit differencing in time is OK (provided soln decays so $\int |\nabla u|^2$ is finite)

- uniqueness via energy method is OK (provided soln decays so $\int |\nabla u|^2$ is finite)

- uniqueness via max prin requires additional arg't, to rule out possibility of max being achieved as $|x| \to \infty$.

<u>Soln formula</u>, for initial-value pblm in $\mathbb{R}^n$: if u_0 is cont's and unif bdd on $\mathbb{R}^n$ then soln is

$$u(x,t) = \int_{\mathbb{R}^n} \Phi(y-x,t)\, u_0(y)\, dy$$

with

$$\Phi(z,t) = \frac{1}{(4\pi t)^{n/2}}\, e^{-|z|^2/4t}$$ ("fund soln").

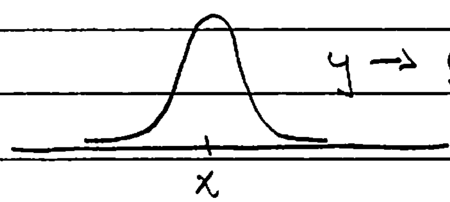
<u>Sketch of pf</u>:

a) $\Phi_t - \Delta\Phi = 0$ (easy to check directly)

b) $\int_{\mathbb{R}^n} \Phi(z,t)\, dz = 1$ for any t

(easy to check, using that $\int_{-\infty}^{+\infty} e^{-x^2} dx = \sqrt{\pi}$)

c) as $t \to 0$, $\Phi(y-x,t)$ is highly concentrated near $y = x$



$y \to \Phi(y-x,t)$ is a Gaussian with variance $\sim t$

d) $\int \Phi(x-y,t)\, u_0(y)\, dy = \int_{|y-x|<\varepsilon} \text{---} + \int_{|y-x|\geq\varepsilon} \text{---}$.

As $t \to 0$,

1st term $\approx u_0(x)$ using cont'y of u_0

2nd term ≈ 0, using bddness of u_0.

Note: Since Φ decays exponentially as $z \to \infty$ (with t held fixed), soln formula does not require that u_0 decay at ∞. In fact it can grow rather rapidly.

How could we have guessed the solution formula?

method 1: we saw in lecture 1 that finite difference approxn of heat eqn is assoc to a coin-flipping random walk. So continuum should be assoc to a suitable limit of random walks.

Probabilistic interpretn of soln formula:

$\Phi(y-x,t)$ = prob of being at y at time t, for a walker who starts at x at time 0.

Since central limit thm says distrn of positions after many coin-flips is asympt. Gaussian, we certainly expect $\Phi(z,t)$ to be Gaussian wrt z. (Dependence on t can be deduced from reqt that $\int \Phi(z,t)\,dz = 1$.)

method 2: Laplacian does have eigenfns in $\mathbb{R}^n$, namely $e^{i\xi\cdot x}$ for any $\xi\in\mathbb{R}^n$. They're not in L^2, and they form a continuum rather than a countable family. But the analogue of our basic expansion in $L^2(\Omega)$ is the Fourier transform:

$$f(x) = (2\pi)^{-n/2}\int e^{i x\cdot\xi}\hat{f}(\xi)\,d\xi$$

$$\hat{f}(\xi) = (2\pi)^{-n/2}\int e^{-i x\cdot\xi} f(x)\,dx$$

Since $\Delta(e^{i x\cdot\xi}) = -|\xi|^2 e^{i x\cdot\xi}$, soln of initial value pblm for heat eqn should be

$$u(x,t) = (2\pi)^{-n/2}\int e^{-|\xi|^2 t}\,\hat{u}_0(\xi)\,e^{i x\cdot\xi}\,d\xi$$

$$= (2\pi)^{-n}\int e^{-|\xi|^2 t}\,e^{i(x\cdot\xi)}\,e^{-i(y\cdot\xi)}\,u_0(y)\,dy\,d\xi$$

$$= \int \Phi(x-y,t)\,u_0(y)\,dy$$

with $\Phi(z,t) = (2\pi)^{-n}\int e^{-|\xi|^2 t + i(z\cdot\xi)}\,d\xi$.

Doing the integral: if $\eta = \xi\sqrt{t} - \frac{iz}{2\sqrt{t}}$ then

$$\Phi(z,t) = (2\pi)^{-n} \int e^{-|\eta|^2} e^{-|z|^2/4t}\, t^{-n/2}\, d\eta$$

$$= (2\pi)^{-n} e^{-|z|^2/4t}\, t^{-n/2} \underbrace{\int e^{-|\eta|^2} d\eta}_{\pi^{n/2}}.$$

$$= (4\pi t)^{-n/2} e^{-|z|^2/4t}$$

as expected.

Having $u(x,t)$ as an "explicit" integral involving u_0 is convenient. Can our "solution formula" for bdd domains be expressed this way? Yes indeed. For example, for

$$u_t - \Delta u = 0 \quad \text{in } \Omega \quad (\text{bounded}).$$
$$u = 0 \quad \text{at } \partial\Omega$$
$$u = u_0 \quad \text{at } t = 0$$

$$u(x,t) = \int_\Omega G(x,y;t)\, u_0(y)\, dy$$

where

$$G(x,y;t) = \sum_{n=1}^{\infty} e^{-\lambda_n t} \varphi_n(x)\varphi_n(y)$$

This is just a rewrite of our soln formula; here φ_n are an orthonormal basis of eigenfns, $-\Delta\varphi_n = \lambda_n \varphi_n$ with $\varphi_n = 0$ at $\partial\Omega$; note that G is symmetric in $x + y$, but no longer a fn of $x-y$.

$G(x,y;t)$ is called the "Green's fn" for the heat eqn. (There's an entirely analogous discn of course for Neumann bc $\partial u/\partial\nu = 0$ at $\partial\Omega$, using eigenfns of $-\Delta$ with $\partial\varphi_n/\partial\nu = 0$.)

Is soln unique?

- yes, if we assume $\nabla u \to 0$ at ∞ fast enough to permit this "energy arg't":

$$\frac{d}{dt}\int_{\mathbb{R}^n} \tfrac{1}{2}|\nabla u|^2 = \int_{\mathbb{R}^n} \langle \nabla u, \nabla u_t \rangle = -\int_{\mathbb{R}^n} \Delta u \cdot u_t$$

If $u_t = \Delta u$ then RHS becomes $-\int u_t^2 \leq 0$. So $u_t - \Delta u = 0$ and $u = 0$ initially + preceding calc justified $\Rightarrow u = 0$. (Could also argue using $\frac{d}{dt}\int_{\mathbb{R}^n} u^2$ as we did in Lecture 1.)

- no, in general: if we permit extreme growth as $|x| \to \infty$ there is a nonzero soln of $u_t - \Delta u = 0$ with initial data 0 (see F. John for an example).

Actually, max principle gives uniqueness with very mild cond on growth as $|x| \to \infty$

<u>Max prin for heat eqn in $\mathbb{R}^n$</u>: if $u_t - \Delta u \leq 0$ for $x \in \mathbb{R}^n$, $0 < t < T$ and also

$$u(x,t) \leq M e^{a|x|^2}$$

for some $M, a > 0$, then for $0 < t < T$ we have

$$u(x,t) \leq \max \left[u \big|_{t=0} \right].$$

<u>Pf</u>: May suppose T is small if necessary (otherwise argue in discrete time steps). We'll specify the smallness condn later.

Consider $u_\varepsilon(x,t) = u(x,t) - \varepsilon \underbrace{(4\pi(T-t))^{-n/2} e^{|x|^2/4(T-t)}}$

Coefft of ε is $\Phi(ix, T-t)$, so it solves the heat eqn! So $\partial_t u_\varepsilon - \Delta u_\varepsilon \leq 0$.

Now look at $B_\rho \times (0,T)$, ρ sufficiently large.

We know

$$\max_{\substack{|x|\le\rho \\ 0\le t\le T}} u_\varepsilon \text{ is achieved at } t=0 \text{ or else at } |x|=\rho.$$

But at $|x|=\rho$,

$$u_\varepsilon \le M e^{a\rho^2} - \varepsilon (4\pi T)^{-n/2} e^{\rho^2/4T}$$

Let's assume $\frac{1}{4T} > a$. Then ε term wins as $\rho \to \infty$, i.e. restr of u_ε to $|x|=\rho$ approaches $-\infty$ as $\rho\to\infty$. Then using suffitly large ρ, we get

$$\max_{\substack{x\in\mathbb{R}^n \\ 0\le t\le T}} u_\varepsilon = \max_{\substack{x\in\mathbb{R}^n \\ t=0}} u_\varepsilon \le \max_{x\in\mathbb{R}^n} u\Big|_{t=0}$$

Now let $\varepsilon\to 0$.

[Note: there is of course an analogous min principle, using a symmetric arg^t or applying the max princ to $-u$]

What about heat eqn in a half-space? For Dir bc ($u=0$) or Neumann bc ($\partial u/\partial \nu = 0$), soln is easy to write down by reflection.

Do it in 1D for simplicity:

$$u_t - u_{xx} = 0 \text{ for } x > 0, \quad u = 0 \text{ at } x = 0$$

$$\Rightarrow u(x,t) = \int_0^\infty G(x,y,t)\, u_0(y)\, dy$$

where $G(x,y,t)$ (the Green's fn for a half space with Dir bc) is

$$(*) \qquad G(x,y;t) = \Phi(x-y,t) - \Phi(x+y,t)$$

Pf: extend u_0 by odd reflection

$$\tilde{u}_0(x) = \begin{cases} u_0(x) & x > 0 \\ -u_0(-x) & x < 0 \end{cases}$$

then apply whole-space soln formula to $\tilde{u}_0$. Resulting soln is odd in x, so it vanishes at $x = 0$.

The case of a Neumann bc is similar:

$$u_t - u_{xx} = 0 \text{ for } x > 0, \quad u_x = 0 \text{ at } x = 0$$

$$\Rightarrow u(x,t) = \int_0^\infty N(x,y,t)\, u_0(y)\, dy$$

where $N(x,y,t) = \Phi(x-y,t) + \Phi(x+y,t)$.

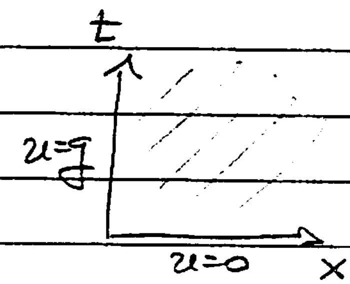
Pf: extend u_0 by even reflection

$$\tilde{u}_0(x) = \begin{cases} u_0(x) & x>0 \\ u_0(-x) & x<0 \end{cases}$$

Then apply whole-space soln formula to $\tilde{u}_0$. Resulting soln is even in x, ie $u(x,t) = u(-x,t)$. So $u_x = 0$ at $x=0$.

What abt halfspace with inhomogeneous bdry data? To fix ideas, let's discuss

$$u_t - u_{xx} = 0 \text{ for } x>0$$
$$u = g(t) \text{ for } x=0$$
$$u = 0 \text{ at } t=0$$



Soln is: $u(x,t) = \int_0^t \frac{\partial G}{\partial y}(x,0,t-s)\,g(s)\,ds$

where G is the Green's fn of the half-space pbm with homog Dir bc (given by eqn (*) on pg 3.9). After some arithmetic this amounts to

$$u(x,t) = \int_0^t \frac{x}{\sqrt{4\pi}\,(t-s)^{3/2}}\, e^{-x^2/4(t-s)}\, g(s)\,ds.$$

Proof is simple (and same idea works even in multiple dimensions + for bounded domains).
Fix x_0, t_0 + consider

$$v(y,\tau) = G(x_0, y, t_0 - \tau).$$

It solves heat eqn backward in time, with δ-fn singularity at $y = x_0$ at $\tau = t_0$. So

$$\lim_{\tau \uparrow t_0} \int_0^\infty v(y,\tau)\, u(y,\tau)\, dy = u(x_0, t_0)$$

But

$$\frac{d}{ds}\int_0^\infty v(y,s)\, u(y,s)\, dy = \int_0^\infty v_s u + v u_s$$

$$= \int_0^\infty -v_{yy} u + v u_{yy}$$

$$= -v_y u \Big|_0^\infty + v u_y \Big|_0^\infty$$

Now, $v \to 0$ and $v_y \to 0$ as $y \to \infty$, and $v = 0$ at $y = 0$.
So we get

$$\frac{d}{ds}\int_0^\infty v(y,s)\, u(y,s)\, dy = v_y u \Big|_{y=0}$$

Integrating, and using that $u = 0$ at $s = 0$, we get

$$u(x_0, t_0) = \int_0^{t_0} v_y(0,s)\, u(0,s)\, ds$$

which was exactly our assertion.

(Similar arg't works with nonzero Neumann data; then of course we must use the Green's fn assoc to a zero Neumann cond, which we called $N(x,y;t)$ on pg 3.9.)

Nonzero source term? Situation is essentially the same as for bounded domain: sol'n of

$$u_t - \Delta u = f \qquad \text{for } t>0$$
$$u = u_0 \qquad \text{at } t=0$$

is

$$u(\cdot,t) = e^{t\Delta}u_0(\cdot) + \int_0^t e^{(t-s)\Delta} f(\cdot,s)\,ds$$

where now

$e^{t\Delta}\varphi(\cdot)$ = sol'n of heat eqn for time t with initial data φ

= convolution of φ with the fundamental solution $\Phi(z,t)$

Bottom line: some things are easier in $\mathbb{R}^n$ (or for half-spaces) because we have a very explicit soln formula. Some examples:

- soln is instantly C^∞, no matter how bad u_0 might be (also true for bdd domains, but easier to see in $\mathbb{R}^n$) – just differentiate under the integral in

$$u(x,t) = \int \Phi(x-y,t)\, u_0(y)\, dy$$

- besides being ill-posed, soln backward in time may cease to exist at a particular time (again, also true for bdd domains but easy to see in $\mathbb{R}^n$) – just consider final-time data $u(x,T) = \Phi(x,T)$. (Soln backward in t is $\Phi(x,t)$, but ceases to exist at $t=0$.)