

DF - Lecture 13, 12/10/2013

Central topic: Hamilton-Jacobi eqns.

Goals: understand

- Hopf-Lax solution formula for $u_t + H(\nabla u) = 0$;
- link between optimal control and 1st order pde

We'll also touch (briefly) on the notion of "viscosity solution" but we won't have much time for that.

Suggested reading: 2.3.3 and 2.10.3 of Evans (which cover all I'm discussing here, and much more).

Recall our discussion of the role of (small) viscosity in selecting a special ("admissible") solution of Burgers' eqn:

- viscous perturbation is $u_t + uu_x = \varepsilon u_{xx}$, $u|_{t=0} = g$
- integrate once to get $w_t + \frac{1}{2}(w_x)^2 = \varepsilon w_{xx}$, $w|_{t=0} = h$
- change vars by $v = \exp(-w/2\varepsilon) \Rightarrow v$ solves the linear heat eqn $v_t - \varepsilon v_{xx} = 0$
- behaviour of explicit solution formula as $\varepsilon \rightarrow 0$ can be studied using "Laplace asymptotics", leading to $w(x,t) = \min_y \left\{ \frac{|x-y|^2}{2t} + h(y) \right\}$

Today we'll gain a new perspective on this var'ial principle for \mathbb{W} (without Laplace asymptotics).

Our discussion works most naturally for the final-value problem

$$(*) \quad u_t + H(\nabla u) = 0 \text{ for } t < T, \quad u = g \text{ at } t = T,$$

where $x \in \mathbb{R}^n$ and $H: \mathbb{R}^n \rightarrow \mathbb{R}$. We'll need to assume H is convex or H is concave. (If you really want to solve an initial value problem, note that for $\tau = T - t$, $\tilde{u}(x, \tau) = u(x, t)$ solves $\tilde{u}_\tau - H(\nabla \tilde{u}) = 0$ with $\tilde{u} = g$ at $\tau = 0$.)

Optimal control interpretation of (*), for H convex: consider

$$(**) \quad u(x, t) = \max_{\alpha(s)} \left\{ \int_t^T h(\alpha(s)) ds + g(y(T)) \right\}$$

$$\left[\begin{array}{l} \frac{dy}{ds} = \alpha(s) \\ y(t) = x \end{array} \right] \leftarrow \text{"state eqn"}$$

where $h: \mathbb{R}^n \rightarrow \mathbb{R}$ is related to H by

$$H(p) = \max_{a \in \mathbb{R}^n} \{ a \cdot p + h(a) \}$$

or equivalently

$$-h(a) = \max_{p \in \mathbb{R}^n} \{a \cdot p - H(p)\}$$

(example: $h(a) = -\frac{1}{2}|a|^2$, $H(p) = \frac{1}{2}|p|^2$; in general, $-h$ is the "Fenchel transform" of H and vice versa; note that h is concave.)

Claim 1: Since h is concave, an optimal path $y(s)$ should have const velocity, i.e. $\alpha(s)$ should be constant.

Pf: h concave \Rightarrow by Jensen's ineq,

$$h[\text{avg velocity}] \geq \text{average of } h(\text{velocity})$$

$$\text{Now, avg velocity} = \frac{1}{T-t} \int_t^T \frac{dy}{ds} ds = \frac{y(T) - y(t)}{T-t}$$

$$\begin{aligned} \text{So } \int_t^T h(\alpha(s)) ds &= (T-t) \cdot \text{avg of } h(\text{velocity}) \\ &\leq (T-t) \cdot h(\text{avg velocity}) \\ &= (T-t) h\left(\frac{y(T) - y(t)}{T-t}\right) \end{aligned}$$

= value assoc to const-velocity path with $\alpha \equiv \frac{y(T) - y(t)}{T-t}$

QED

Conclusion: The function u defined by (**)
has the simpler characterization

$$u(x, t) = \max_{z \in \mathbb{R}^n} \left\{ (T-t) h\left(\frac{z-x}{T-t}\right) + g(z) \right\}$$

This is the "Hopf-Lax solution formula" for the final-time problem (*).

Note: The preceding was for convex H + final-value problem;

- as explained above, initial value problem for $u_t - H(\nabla u) = 0$ with H convex is equivalent
- for the final-value problem with concave H , must use min instead of max;

$$u(x, t) = \min_{\alpha(\cdot)} \left\{ \int_t^T h(\alpha(s)) ds + g(y(T)) \right\}$$

$$\left[\begin{array}{l} \frac{dy}{ds} = \alpha(s) \\ y(t) = x \end{array} \right]$$

$$\text{where } H(p) = \min_{a \in \mathbb{R}^n} \{ a \cdot p + h(a) \}$$

$$-h(a) = \min_{p \in \mathbb{R}^n} \{ a \cdot p - H(p) \}$$

(so h is convex; example: $H(p) = \frac{1}{2}|p|^2$, $h(a) = \frac{1}{2}|a|^2$)

- For the initial value problem $u_t - H(\nabla u) = 0$ with H concave, we use the last bullet with $\tau = T - t$.

End of digression, we return now to (**).

Claim 2: $u(x, t)$, defined by (**), solves (at least formally) the pde

$$\begin{aligned} u_t + H(\nabla u) &= 0 & t < T \\ u &= g & t = T \end{aligned}$$

provided $H(p) = \max_{a \in \mathbb{R}^n} \{a \cdot p + h(a)\}$

Explanation: Consider paths $x = t \frac{dx}{dt} = a$ (constant) briefly, for time Δt . Suppose that as $\Delta t \rightarrow 0$ they're suitable for optz. Then

$$u(x, t) \approx \max_a \{ h(a) \Delta t + u(x + a \Delta t, t + \Delta t) \}.$$

Now suppose u is smooth enough to justify the following Taylor-expansion-based argt:

$$u(x, t) \approx \max_a \left\{ h(a) \Delta t + u(x, t) + a \cdot \nabla u(x, t) \Delta t + \frac{1}{2} \nabla^2 u(x, t) \Delta t^2 \right\}$$

(up to errors of order $(\Delta t)^2$).

Cancel $u(x, t)$ from both sides + simplify \Rightarrow

$$0 \approx \max_a \{h(a) + a \cdot \nabla u\} + u_t$$

ie $0 = u_t + H(\nabla u)$ with H defined as given above.

Example: $h(a) = -\frac{1}{2}|a|^2$ (concave) $\Rightarrow H(a) = \frac{1}{2}|a|^2$ (convex)
and we get "soln formula" for

(~~***~~) final-value prob $u_t + \frac{1}{2}|\nabla u|^2 = 0$, $u(x, T) = g(x)$

in the form $u(x, t) = \max_z \left\{ g(z) - \frac{|z-x|^2}{2(T-t)} \right\}$

Observations:

a) HJ eqn has infinitely many cc solns, only one of which agrees with our "soln formula"

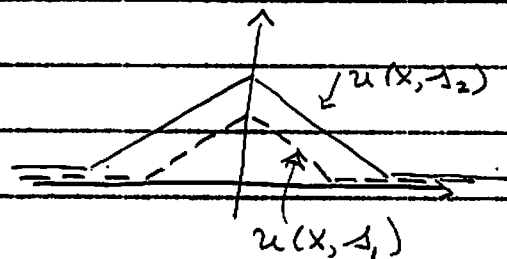
b) u given by soln formula can have discontinuous derivatives (graph can have sharp valleys, when viewed as fn of x)

c) u given by soln formula cannot have sharp peaks

About (a): Consider $g=0$, and 1D version of (***)
Then soln formula gives $u=0$. But

$$u(x,t) = \begin{cases} \frac{1}{2}(T-t) - |x| & \text{if } |x| \leq \frac{1}{2}(T-t) \\ 0 & \text{otherwise} \end{cases}$$

is also an ac soln



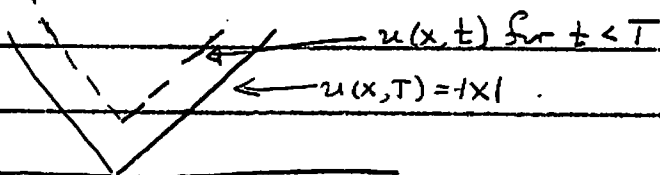
(Also: this artificial "bump" could be located anywhere - it need not be at $x=0$.)

$$t_2 < t_1 < T$$

About (b): Consider the same example (1D version of (***)) but with $g(y) = |y|$. Then soln formula is easy to evaluate; best \tilde{z} is $x + (T-t)^+ |x|$, yielding

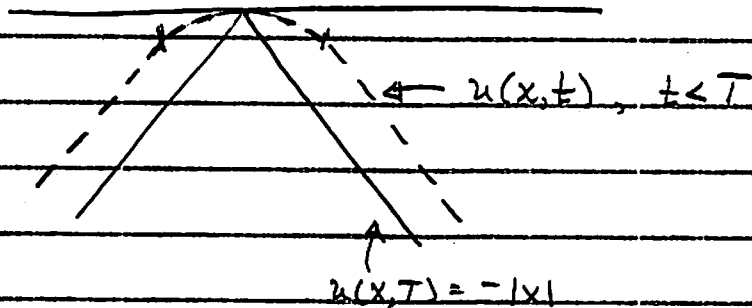
$$u(x,t) = \frac{T-t}{2} + |x|$$

(best \tilde{z} is nonunique at $x=0$, where u is singular)



Concerning (c): Still considering 1D version of (***)
 try $g(y) = -|y|$. Then best z is $z = (T-t)z/|x|$ if $|x| > T-t$
 but $z=0$ otherwise. Get:

$$u(x,t) = \begin{cases} \frac{T-t}{2} - |x| & |x| > T-t \\ \frac{-|x|^2}{2(T-t)} & |x| \leq T-t \end{cases}$$



Evidently: HT eqn $u_t + H(\nabla u) = 0$ has problem of nonuniqueness, like scalar cons laws. We need a selection criterion.

Fact: small viscosity and the optimal control interpretation make the same selection. Proof is in Evans 3.10.3 + lies beyond scope of this class. But let's show that as $\varepsilon \rightarrow 0$, soln of

$$u_t^\varepsilon + \frac{1}{2}(u_x^\varepsilon)^2 + \varepsilon u_{xx} = 0 \quad \text{for } t < T$$

$$u^\varepsilon = g \quad \text{at } t = T$$

cannot have a sharp peak (cf observation (c) above). If it did then u^ε would have a rounded peak, & at the top

$$\begin{aligned} u_x^\varepsilon &= 0 && (u^\varepsilon \text{ is smooth for } \varepsilon > 0!) \\ u_{xx}^\varepsilon &\leq 0 \end{aligned}$$

$$\Rightarrow u_{tt}^\varepsilon \geq 0 \text{ at the peak.}$$

But if $u = \lim_{\varepsilon \rightarrow 0} u^\varepsilon$ has a "sharp" peak (discontinuity in u_x) then $u_{tt} = -\frac{1}{2} u_x^2 < 0$ at the peak. Contradiction.

Theory of "viscosity solns" gives a characterization of soln of

$$\begin{aligned} u_t^\varepsilon + H(\nabla u^\varepsilon) + \varepsilon \Delta u^\varepsilon &= 0 && t < T \\ u^\varepsilon &= g && \text{at } t = T \end{aligned}$$

in limit $\varepsilon \rightarrow 0$ (without actually introducing ε , and without a soln formula - for example it works the same even if H is neither convex nor concave). See Evans.

Preceding descr is not easy to visualize. Here is one that's easier: fix a "target set" $G \subset \mathbb{R}^n$. For $x \notin G$, consider "min arrival time"

13.10

$$z(x, t) = \min \left\{ \text{1st time } y(s) \text{ arrives at } G \right\}$$

$$\left[\begin{array}{l} y(0) = x \\ \frac{dy}{ds} = f(y(s), u(s)) \end{array} \right] \text{ "state eqn"}$$

$$u(s) \in A \quad \leftarrow \text{"admissible controls"}$$

Formal HT eqn (new time indep) is

$$z(x) \approx \min_{a \in A} \left\{ \Delta t + z(x + f(x, a) \Delta t) \right\}$$

$$\approx \min_{a \in A} \left\{ \Delta t + z(x) + f(x, a) \cdot \nabla z(x) \cdot \Delta t \right\}$$

$$\Rightarrow 0 = \min_{a \in A} \left\{ f(x, a) \cdot \nabla z(x) \right\} + 1$$

Special case: use state eqn $dy/ds = f(y, u)$
and impose constraint $|u(s)| \leq \phi(y(s))$
Then preceding calcn gives

$$\min_{|a| \leq \phi(x)} \nabla z \cdot a + 1 = 0$$

i.e.

$$-\phi(x) |\nabla z| + 1 = 0$$

Thus: $|\nabla z| = 1/\phi(x)$ for $x \notin G$.
 $z = 0$ at G .

13.11

Special case $\epsilon \equiv 1$ is familiar "ekoval epn"

$$|v| = 1 \quad \text{for } x \notin G.$$

$$v = 0 \quad \text{at } G$$

In this case the best paths are straight lines
(which are characteristics in the pole!).