PDE - Lecture 12 - 12/3/2013

We spent most of the semester on very special eqns ($\Delta u = 0$, $u_t = u_{xx}$, $u_{tt} = \Delta u$, etc). They are examples of elliptic, parabolic, + hyperbolic eqns. It's time to discuss what this means.

Main goals:

a) general classifn of 2nd order pde in two vars

b) discuss, for linear pde (in any space dimn and any order) the notion of a "noncharacteristic surface"

c) link this discn (briefly) to the Cauchy-Kowalewsky Thm.

General classifn of 2nd order pde in 2 vars:

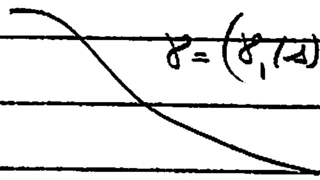
consider

$$a(x,y)\, u_{xx} + 2b(x,y)\, u_{xy} + c(x,y)\, u_{yy}$$
$$+ d(x,y)\, u_x + e(x,y)\, u_y + f(x,y)\, u = g(x,y),$$

It is
elliptic if $b^2 - ac < 0$
hyperbolic if $b^2 - ac > 0$
parabolic if $b^2 - ac = 0$

(Note that if a, b, c are fns of $x+y$ then the type can be different at different points.)

Question: when can we specify u and $\frac{\partial u}{\partial n}$ along a curve γ in the x-y plane and thereby determine (at least formally) the fn u nearby?



$\gamma = (\gamma_1(s), \gamma_2(s))$

Idea: use eqn to solve for 2nd derivs along γ, if possible.

Know $\frac{du_x}{ds}$ + $\frac{du_y}{ds}$ (since $u + \frac{\partial u}{\partial n}$ determine ∇u along γ)

By chain rule

$$\frac{du_x}{ds} = u_{xx}\gamma_1'(s) + u_{xy}\gamma_2'(s)$$

$$\frac{du_y}{ds} = u_{xy}\gamma_1'(s) + u_{yy}\gamma_2'(s)$$

So we have 3 lin eqns in 3 unknowns u_{xx}, u_{xy}, u_{yy} at each pt of γ:

$$a u_{xx} + 2b u_{xy} + c u_{yy} = \text{known}$$
$$\gamma_1' u_{xx} + \gamma_2' u_{xy} = \text{known}$$
$$\gamma_1' u_{xy} + \gamma_2' u_{yy} = \text{known}$$

These determine u_{xx}, u_{xy}, u_{yy} if the determinant is nonzero, ie if

$$0 \neq \det \begin{pmatrix} a & 2b & c \\ \gamma_1' & \gamma_2' & 0 \\ 0 & \gamma_1' & \gamma_2' \end{pmatrix} = a(\gamma_2')^2 + c(\gamma_1')^2 - 2b\gamma_1'\gamma_2'$$

Note: normal to γ is in dirn $(\gamma_2', -\gamma_1')$ so condn is

$$a n_1^2 + 2b n_1 n_2 + c n_2^2 \neq 0 \qquad n = \vec{n}_\gamma$$

A curve where this holds is called noncharacteristic. One where it fails (ptwise) is a characteristic.

Evidently: elliptic $\Leftrightarrow$ every curve is noncharacteristic
hyperbolic $\Leftrightarrow$ two distinct families of characteristic curves
parabolic $\Leftrightarrow$ just one characteristic curve

Notes:

- in finding u_{xx}, u_{xy}, u_{yy} what we mainly did was find 2nd derivs in normal dirn (since

$\partial_{tan} u$, $\partial_{tan}\partial_{tan} u$, $\partial_{tan}\partial_{nor} u$ are easy to get from data by diff'n)

- for wave eqn $u_{tt} - u_{xx} = 0$, the characteristics are of course lines $x-t = \text{const}$ + $x+t=\text{const}$.

- for heat eqn $u_t - u_{xx} = 0$ the characteristic surfaces are $t=\text{const}$. (And indeed: we cannot specify both u + u_t independently at $t=0$!!)

- since this classifn depends only on the quadr form $\begin{pmatrix} a & b \\ b & c \end{pmatrix}$ evaluated at $\vec{n}_\gamma$, one easily verifies that it doesn't depend on the choice of coord system.

For a quasilinear eqn

$$a(x,y,u,\nabla u)\,u_{xx} + 2b(x,y,u,\nabla u)\,u_{xy} + c(x,y,u,\nabla u)\,u_{yy} = F(x,y,u,\nabla u)$$

we can use the same classifn by simply viewing $a(x,y,u,\nabla u) = \bar{a}(x,y)$. Of course the type now depends on the particular soln u being considered.

For 2nd order eqns in ≥ 3 vars the classifn is less complete. Consider a linear eqn

$$(*) \quad \sum a_{ij}(\vec{x}) \frac{\partial^2 u}{\partial x_i \partial x_j} + \text{lower order} = 0$$

We call it

elliptic if $a_{ij}(x)$ is pos definite quadratic form

hyperbolic if $a_{ij}(x)$ has one neg eigenvalue + all other eigenvalues are positive

parabolic if $a_{ij}(x)$ has one zero eigenvalue + all others pos.

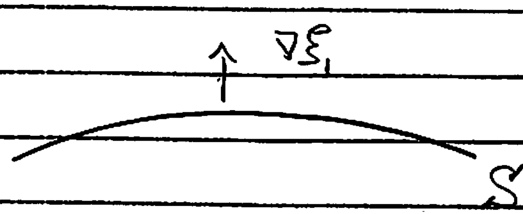
But clearly this doesn't exhaust all possible cases.

Let's explore analogue for $(*)$ of what we did in 2D. Suppose u + $\partial u/\partial n$ are specified on a hypersurface S. When does this permit us (using the eqn) to (formally) find $D^2 u$ on S?

It's convenient (+ no loss of generality) to suppose S is (locally) of form

$$S = \{ \xi_1(x_1, \dots, x_n) = \text{const} \}$$

+ we may introduce further coords $\xi_2, \dots, \xi_n$ st $\nabla\xi_2, \dots, \nabla\xi_n$ are tangent to S (all this is local!).



Let's change vars in eqn:

$$\frac{\partial u}{\partial x_i} = \sum_k{}' \frac{\partial u}{\partial \xi_k} \frac{\partial \xi_k}{\partial x_i}$$

$$\frac{\partial^2 u}{\partial x_i \partial x_j} = \sum_{k,l}{}' \frac{\partial^2 u}{\partial \xi_k \partial \xi_l} \frac{\partial \xi_k}{\partial x_i} \frac{\partial \xi_l}{\partial x_j} + (1^{st} \text{ order terms in } u)$$

So pde says $\sum{}' A_{kl} \frac{\partial^2 u}{\partial \xi_k \partial \xi_l}$ = terms of 1st order or less

where

$$A_{kl} = \sum{}' a_{ij} \frac{\partial \xi_k}{\partial x_i} \frac{\partial \xi_l}{\partial x_j}$$

Now, Cauchy data (u and $\partial u / \partial n$) determines

$\frac{\partial^2 u}{\partial \xi_i \partial \xi_j}$ for all pairs i, j except $i=1, j=1$.

Key question: does eqn determine $\frac{\partial^2 u}{\partial \xi_1^2}$?

Ans is yes $\iff A_{11} \neq 0$

$$\iff \sum a_{ij} \frac{\partial \xi_1}{\partial x_i} \frac{\partial \xi_1}{\partial x_j} \neq 0$$

$$\iff \sum a_{ij} n_i n_j \neq 0 \quad \text{, where } \vec{n} = \vec{n}_\xi .$$

Such a surface is noncharacteristic.

Notes:

- if eqn is elliptic then every surface is noncharacteristic
- for wave eqn $u_{tt} - \Delta u = 0$ in $\mathbb{R}^n$, surface is noncharacteristic if $n = (n_x, n_t) \in \mathbb{R}^{n+1}$ has $|n_x|^2 - n_t^2 \neq 0$

Note that our terminology is consistent with "method of characteristics" for 1st order eqns

In fact: for 1st order eqn, analogous question is this: if we specify u along S, does eqn give us $\partial u/\partial n$ (thus formally determining u nearby)?

Ans was: for $\sum a_i \frac{\partial u}{\partial x_i} = 0$, $\frac{\partial u}{\partial n}$ is determined if $\vec{a} \cdot \vec{n}_S \neq 0$. So characteristics must be transverse to S.

Thus, for 1st order case, a "characteristic surface" is one swept out by characteristics.

I focused above only on whether knowing u + $\partial u/\partial n$ on S determines $D^2 u|_S$. But if surface is noncharacteristic, one can see by essentially the same method (differentiating the eqn) that the entire Taylor expansion of u is determined by knowing u + $\partial u/\partial n$ on S. So there can be at most one soln of the Cauchy pblm.

For higher order eqns the same ideas can be used. If eqn is $\sum_{|\alpha|=m} a_\alpha D^\alpha u$ = lower order terms, then

$$S \text{ is nonchar} \iff \sum_{|\alpha|=m} a_\alpha n_{\alpha_1} \cdots n_{\alpha_m} \neq 0.$$

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We've seen that if S is nonchar then Cauchy data formally determines soln u nearby. But does this mean there is a soln?

Ans is yes, locally, if everything in sight is analytic [This is the Cauchy-Kowalevsky thm]

Ans is no, in general, without analyticity; for example if eqn is $\Delta u = 0$ then existence of a local soln forces data to be analytic (harmonic fns aren't just C^∞, they are actually real analytic)