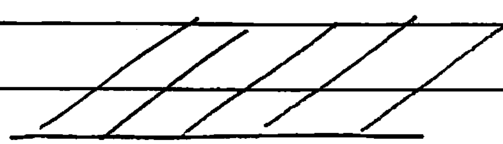
PDE – Lecture 11 – 11/26/2013

Today: a brief discn about numerical soln of conservation laws; then we'll turn to a general discn of the method of characteristics for solving 1st order pde's.

Numerical soln of scalar cons laws. (My source: Chap 4 of J. Sethian's book "Level Set Methods", Camb Univ Press – a conveniently elementary + brief introduction.)

Consider 1st the linear eqn $u_t + u_x = 0$, whose characteristics have slope 1 (the gen soln is $u = f(x-t)$)



The natural discretization is the one whose numerical domain of dependence mimics that of the pde: if $x_j = j\Delta x$ and $u_j(t) = u(x_j, t)$

$$(*) \qquad \frac{u_j(t+\Delta t) - u_j(t)}{\Delta t} + \underbrace{\frac{u_j(t) - u_{j-1}(t)}{\Delta x}}_{\substack{\text{spatial discretization should be} \\ \text{"upwind," i.e. should respect the} \\ \text{direction from which info arrives}}} = 0$$

If instead we were to use, say, $\frac{u_{j+1}(t) - u_j(t)}{\Delta x}$ in spatial discretization, we would violate the rule that numerical domain of dependence should contain the math'l domain of dependence. (This rule also tells us that for (*) to work we need $\Delta t / \Delta x \leq 1$.)

In nonlinear setting $u_t + (F(u))_x = 0$ we have the new feature that numerical soln must produce shocks (at least approximately) when they're needed, + fans when soln uses them instead.

Simplest idea combines "upwind differencing" with "artificial viscosity;" for Burgers' eqn $u_t + u u_x = 0$ this would suggest

$$\frac{u_j(t+\Delta t) - u_j(t)}{\Delta t} + u_j(t) \cdot \left\{ \begin{array}{ll} \frac{u_j(t) - u_{j-1}(t)}{\Delta x} & \text{if } u_j(t) > 0 \\ \frac{u_{j+1}(t) - u_j(t)}{\Delta x} & \text{if } u_j(t) < 0 \end{array} \right\}$$

$$= \varepsilon \frac{u_{j+1}(t) + u_{j-1}(t) - 2u_j(t)}{(\Delta x)^2}$$

with $\varepsilon > 0$ (small) and $\Delta t / (\Delta x)^2 \leq \frac{1}{2\varepsilon}$.

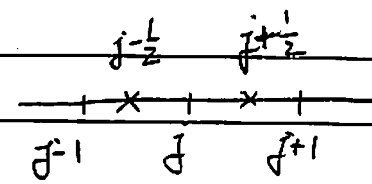
But: This is clunky. It smooths out shocks + makes little use of structure of eqn.

Better idea: use use a scheme that's in <u>conservation form</u>

$$(**) \qquad \frac{u_j(t+\Delta t) - u_j(t)}{\Delta t} + \frac{F_{j+1/2} - F_{j-1/2}}{\Delta X} = 0$$

where the "numerical flux" is some well-chosen function of the nearest values of u

$$F_{j+1/2} = f(u_j, u_{j+1})$$



$$F_{j-1/2} = f(u_{j-1}, u_j)$$

Rewriting (**) in the form

$$u_j(t+\Delta t) = W(u_{j-1}(t), u_j(t), u_{j+1}(t)),$$

The scheme assoc to a given choice of f is said to be <u>monotone</u> if W is a nondecreasing fn of each of its arguments.

<u>Remarkable fact</u>: if W is monotone then (**) produces the "entropy" wk soln, ie it gets the

admissibility of shocks exactly right.

Simple example of such a scheme is Lax-Friedrichs, which is

$$u_j^{n+1}(t) = \frac{1}{2}\left[u_{j-1}(t) + u_{j+1}(t)\right] - \frac{\Delta t}{2\Delta X}\left[F(u_{j+1}(t)) - F(u_{j-1}(t))\right].$$

It is monotone provided $\frac{\Delta t}{\Delta X} F'(u) < 1$, and it can be expressed in cons form using

$$f(u_j, u_{j+1}) = \frac{-\Delta X}{2\Delta t}(u_{j+1} - u_j) + \frac{1}{2}\left[F(u_{j+1}) + F(u_j)\right],$$

so it can be written as

$$\frac{u_j(t+\Delta t) - u_j(t)}{\Delta t} + \frac{F(u_{j+1}) + F(u_{j-1})}{2\Delta X} = \frac{(\Delta X)^2}{2\Delta t}\,\frac{u_{j+1}(t) + u_{j-1}(t) - 2u_j(t)}{(\Delta X)^2}$$

ie it uses "artificial viscosity" $(\Delta X)^2/2\Delta t$. Since the stability cond requires $\Delta t \sim \Delta X$, a "shock" will typically be a few gridpoints wide.

New topic: method of characteristics, for 1st order pde's. The overall goal is to generalize what we achieved for scalar cons laws, where soln of $u_t + c(u)u_x = 0$ was seen to be constant along a line

$\partial x / \partial t$ = const (provided soln is smooth enough).
More general goal: reduce soln of a (1st order) pde to soln of (well-chosen) ode's.

Almost every pde book includes some version of this, often near the beginning. See eg Guenther + Lee chap 2 + F. John Chap 1.

Warning: The word "characteristic" has two related but different uses:

a) in the "method of characteristics" it refers to the soln of the ODE's to which we reduce the pde (eg for scalar cons laws, the characteristics are straight lines in (x, t) space)

b) when considering 2nd order eqns, one can ask whether specifying Cauchy data on a hypersurface determines a unique soln nearby. Ans depends on tangent plane of surface (+ structure of eqn), or equivalently, on the normal vector to the surface. The bad normals are the "characteristic directions". Examples:

i) for $u_{tt} - u_{xx} = 0$, the char. dirns turn out to be $(1,1)$ and $(1,-1)$

ii) for $\Delta u = 0$, there are no char dirns.

We'll come back to this if $\exists$ time (it's closely connected to the Cauchy-Kowalevsky thm).

These 2 uses are different, but consistent: for quasilinear 1st order eqns, specifying u on a curve in space determines the soln nearby if the curve is "noncharacteristic" in a sense to be explained below.

Enough overview, let's get to work. We'll discuss space dim 2 only (it simplifies the notation; going to $\mathbb{R}^n$ requires no further ideas beyond those we'll see in $\mathbb{R}^2$.)

Linear case first:

$$a(x,y)\, u_x + b(x,y)\, u_y = c(x,y)\, u + d(x,y).$$

Eqn reduces to ODE along $x = x(s)$, $y = y(s)$ if

$$dx/ds = a(x,y), \quad dy/ds = b(x,y),$$

since then

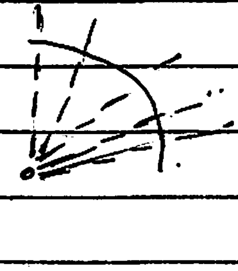
$$\frac{d}{ds} u(x(s),y(s)) = c(x(s),y(s))\,u(x(s),y(s)) + d(x(s),y(s))$$

Geometrically: $\xi = (a,b)$ is a vector field. Eqn says

$$\xi \cdot \nabla u = cu + d.$$

Appropriate "initial data": specify u on a surface that's transverse to the vector field.

Local existence is clear. Global existence can fail: eg if $xu_x + yu_y = 0$ with u = specified data on $x^2+y^2=1$, then we get u = const along each ray thru 0, but it won't (usually) extend smoothly across 0



Next: quasilinear case, ie

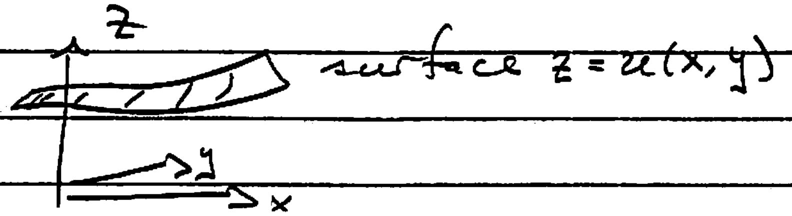
$$a(x,y,u)\,u_x + b(x,y,u)\,u_y = c(x,y,u)$$

Similar method works, but now the eqn of the curve depends on u. We must solve

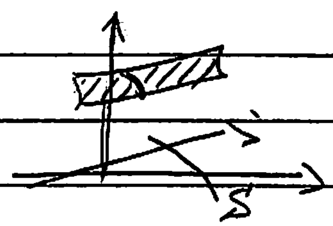
$$\frac{dx}{ds} = a(x,y,z) \qquad \frac{dy}{ds} = b(x,y,z)$$

$$\frac{dz}{ds} = c(x,y,z)$$

(an autonomous system of ODE's). Along such a curve, $z = u(x,y)$ where u is soln of the pde.



Thus: to get a local soln of pde in $\mathbb{R}^2$, we can specify values of u along a curve $S \subset \mathbb{R}^2$ then solve the ODE written above (using the given data for initial values of z). Method works provided $[a(x,y,u), b(x,y,u)]$ is not tangent to S. (This is the condn that S be "noncharacteristic".)



Graph of u is obtained as union of solns of ODE's. Soln is of course strictly local (it may become singular as one tries to extend it).

Finally the nonlinear case: eqn is now

$$F(x, y, u, u_x, u_y) = 0.$$

Use notation $F = F(x, y, u, p, q)$.

Suppose we have a soln u. Consider vector field in $\mathbb{R}^2$:

$$\xi = \left(\frac{\partial F}{\partial p}, \frac{\partial F}{\partial q}\right)\Bigg|_{p = u_x(x,y),\ q = u_y(x,y)}$$

Its integral curves satisfy $\frac{dx}{ds} = \frac{\partial F}{\partial p}$, $\frac{dy}{ds} = \frac{\partial F}{\partial q}$, so along such a curve

$$\frac{d}{ds} u(x(s), y(s)) = u_x \dot{x} + u_y \dot{y}$$

$$= u_x \frac{\partial F}{\partial p} + u_y \frac{\partial F}{\partial q}$$

In quasilinear case $\frac{\partial F}{\partial p}$ + $\frac{\partial F}{\partial q}$ were indep of p + q, but now they aren't, so we need to track them with ODE's. Evidently (since $p = u_x$, $q = u_y$)

$$\frac{dp}{ds} = p_x \dot{x} + p_y \dot{y} = u_{xx} \dot{x} + u_{xy} \dot{y}$$

$$= u_{xx} F_p + u_{xy} F_q$$

$$\frac{dq}{ds} = u_{xy} \dot{x} + u_{yy} \dot{y} = u_{xy} F_p + u_{yy} F_q$$

From eqn,

$$0 = F_x + F_u u_x + F_p u_{xx} + F_q u_{xy}$$

$$0 = F_y + F_u u_y + F_p u_{xy} + F_q u_{yy}$$

so

$$\frac{dp}{ds} = -F_x - F_u u_x = -F_x - F_u p$$

$$\frac{dq}{ds} = -F_y - F_u u_y = -F_y - F_u q$$

Thus we can recover soln to pde restricted to a suitable curve by solving <u>ODE</u> system

$$\dot{x}(s) = F_p$$
$$\dot{y}(s) = F_q$$
$$\dot{u}(s) = p F_p + q F_q$$
$$\dot{p}(s) = -F_x - F_u p$$
$$\dot{q}(s) = -F_y - F_u q$$

(where RHS is eval at $x(s), y(s), u(s), p(s), q(s)$).

When can a (local) soln in (x,y) - space be determined by specifying u along a curve S? Suppose S has form $x=\gamma(\tau)$, $y=\mu(\tau)$ and data are given as $u=\varphi(\tau)$. These clearly give initial data for $x(s,\tau)$, $y(s,\tau)$, $u(s,\tau)$ at $s=0$. What about p + q? Know

$$\varphi_\tau = u_x \gamma_\tau + u_y \mu_\tau = p\gamma_\tau + q\mu_\tau$$

$$0 = F(\gamma, \mu, \varphi, p, q)$$

In quasilinear case this was a linear system for the initial $p+q$ and linear algebra told us whether it had a soln or not. In present case that's less clear (it depends on the form of F)
<u>But</u>: if data admits a unique soln for $p+q$, and further

$$(\gamma_\tau, \mu_\tau) \text{ is not parallel to } (F_p, F_q)$$

(this is the condn that S be "noncharacteristic") Then soln depends smoothly on data (by implicit fn thm) + method determines a unique local soln of the pde.