PDE - Lecture 10 - 11/19/2013

New topic: conservation laws (with Burgers' eqn as the central example).

Recommended reading:

- Guenther + Lee §1.7 (good presentation of the system of cons laws describing an ideal gas)
- Kevorkian §1.7 (on soln of Burgers' eqn via Hopf-Cole transf) and §5.3 (on shocks, fans, wk solns, etc for Burgers' eqn and also some systems of cons laws)
- Evans §3.4.1 (The sections beyond that, §3.4.2 - 3.4.5, go far beyond what we'll cover in this class)

Many physical problems lead to nonlinear, 1st order pde's taking the form of <u>conservation laws</u>. For example: here's the system describing flow of an ideal gas:

$$(*)\begin{cases} \rho_t + \operatorname{div}(\rho \vec{V}) = 0 \\ (\rho V_i)_t + \operatorname{div}(\rho V_i \vec{V}) = -\nabla_i p \qquad i = 1,2,3 \end{cases}$$

where
$$\rho(x,t) = \text{density}$$
$$\vec{V}(x,t) = (V_1, V_2, V_3) = \text{velocity}$$
$$p(x,t) = \text{pressure}$$

and the system is closed by a pressure-density reln such as

$$\frac{p}{p_0} = \left(\frac{\rho}{\rho_0}\right)^\gamma$$

The pde's reflect <u>cons of mass</u> + <u>cons of momentum</u> (see Guenther + Lee §1.7). For example,

cons of mass $\Rightarrow$ for any $D \subset \mathbb{R}^3$

$$\underbrace{\frac{d}{dt}\int_D \rho \, dx}_{\substack{\text{mass} \\ \text{of } D}} = \underbrace{-\int_{\partial D} \rho \vec{V} \cdot n \, dA}_{\substack{\text{rate at} \\ \text{which mass} \\ \text{enters or leaves}}}$$

$$\Rightarrow \int_D \rho_t + \operatorname{div}(\rho \vec{V}) \, dx = 0$$

true for all D $\Rightarrow$ $\rho_t + \operatorname{div}(\rho \vec{V}) = 0$.

We got linear wave eqn earlier by linearizing nonlinear mechanical laws (eg vibrating string), + same thing is possible here: if $\rho/\rho_0 = 1+u$ with u small, and if v is also small, then

$$(\rho/\rho_0)^\lambda = (1+u)^\lambda \approx 1+\lambda u$$

$$[\rho_0(1+u)]_t + \operatorname{div}[\rho_0(1+u)v] = 0 \underset{\text{linearizes to}}{\rightsquigarrow} u_t + \operatorname{div} v = 0$$

$$(\rho_0(1+u)v_i)_t + \cancel{\operatorname{div}(v_i \vec{v}\rho)} + \nabla_i(p_0(1+u)^\lambda) \rightsquigarrow \rho_0 v_t + p_0\lambda\nabla u = 0$$

$$\text{substn} \Rightarrow \quad u_{tt} - \frac{p_0}{\rho_0}\lambda\,\Delta u = 0,$$

The wave eqn with wavespeed $\sqrt{\frac{p_0}{\rho_0}\lambda}$. This is why <u>acoustic waves</u> are described by the linear wave eqn.

<u>But</u>: such a linearization is not always permitted. A familiar example: <u>sonic booms</u> are <u>not</u> described by any linear wave eqn.

Our goal today is to begin to see what can happen in a more fully nonlinear setting. Ideal gases are too complicated; instead we'll focus on <u>scalar</u> <u>conservation</u> <u>laws</u>

$$u_t + (F(u))_x = 0 \qquad t>0$$

$$u = g(x) \qquad \text{at } t=0$$

where now u is scalar-valued and $x \in \mathbb{R}$.

Even in this setting there are physically natural examples:

a) Burgers' eqn $\quad u_t + \frac{1}{2}(u^2)_x = 0$

For smooth solns, eqn is equiv to $u_t + uu_x = 0$. Describes Newtonian motion of non-interacting particles in a 1D continuum:

$u(x,t)$ = velocity of particle that's at posn x at time t

$z(x_0,t)$ = posn of particle originally at x_0

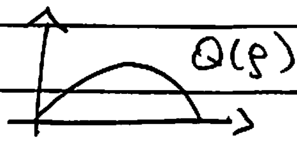
Evidently $\partial_t z = u(z(x_0,t),t)$. Now Newton's laws says $\partial_{tt} z = 0$, which is equiv to $u_t + uu_x = 0$

Burgers' eqn is importnt because it provides a typical model of shock formation. Also since its "viscous perturbation" $u_t + uu_x = \varepsilon u_{xx}$

can be solved explicitly via Hopf-Cole transf.

(b) traffic flow $\rho_t + [Q(\rho)]_x = 0$

Typically



$\rho(x,t)$ = traffic density $\left(\frac{\text{cars}}{\text{meter}}\right)$ at x, t

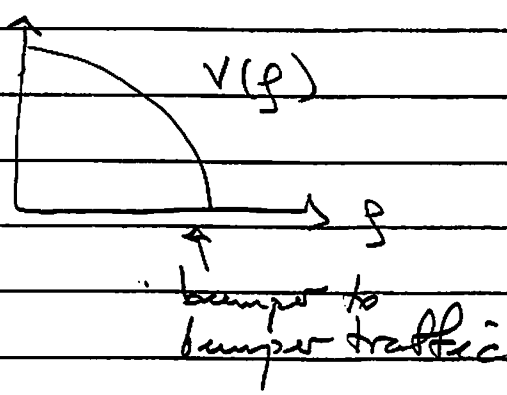
$Q(\rho)$ = rate of traffic flow $\left(\frac{\text{cars}}{\text{hour}}\right)$.

Eqn expresses "cons of cars"

$$\frac{d}{dt}\int_a^b \rho\, dx = -q(b) + q(b)$$

where q = flow rate. (This reln for all $a < b \Rightarrow \rho_t + q_x = 0$.) Reln $q = Q(\rho)$ is a "constitutive law."

Logic for form of $Q(\rho)$: $Q(\rho) = \rho V$ where V = velocity of a typical car. Drivers maintain proper following distance $\Rightarrow V = V(\rho)$, something like



If soln is smooth, we can represent it by the method of characteristics. (This general tool for 1st order eqns is esp. simple in this setting; we'll discuss it in more generality later.) Write eqn as

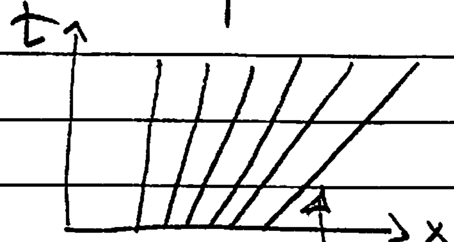
$$u_t + c(u)\, u_x = 0$$

where $c(u) = F'(u)$ (if cons law is $u_t + [F(u)]_x = 0$). Evidently, along a spacetime curve st $dx/dt = c(u)$ we have

$$\frac{d}{dt} u(x(t),t) = u_x \dot{x} + u_t = 0$$

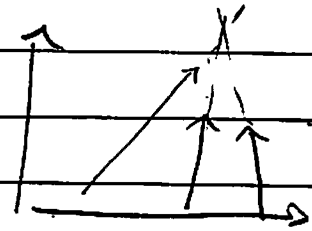
ie u must be constant. These spacetime curves (lines, in this case) are the characteristics of the pde. The soln is especially simple because for this eqn, u is constant along characteristics.

Note: if $c(g(x))$ increases as x increases then the characteristics spread out; no problem there.



slope $\frac{dt}{dx} = \frac{1}{c(g(x))}$, so slope ↓ if $c(g(x))$ ↑

But if $c(g(x))$ decreases as x increases then characteristics will eventually cross (soln develops a shock)



soln cannot stay smooth!

When, exactly, does smooth soln break down? Rephrase preceding discn by

$$u = g(\xi) \quad \text{when} \quad x = \xi + t\,c(g(\xi))$$

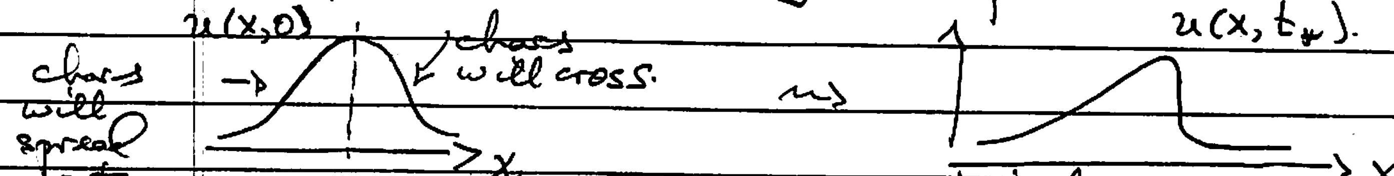
Observe that $\frac{dx}{d\xi} = 1 + t\,\varphi'(\xi)$ where $\varphi(\xi) = c(g(\xi))$.

So map $\xi \to x(t,\xi)$ is single-valued provided $1 + t\varphi'(\xi) > 0$. Breakdown time is

$$t_* = -1/\varphi'(\xi_*)$$

where ξ_* achieves largest $|\varphi'(\xi)|$ among all ξ st $\varphi'(\xi) = c'(g(\xi))\,g'(\xi) < 0$.

Real-space picture, for Burgers' eqn:

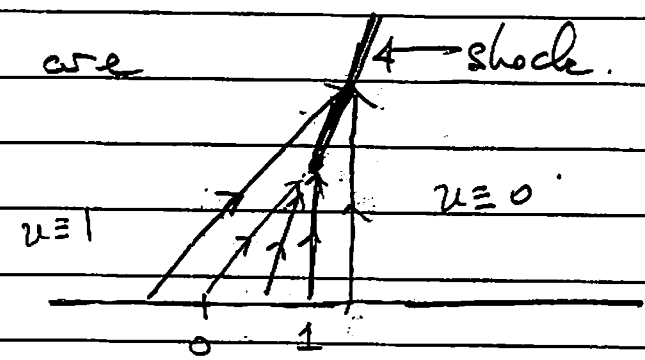


graph is becoming multi-valued

More analytical example, for Burgers' eqn (Evans' example 1 in §3.4.1):

if $u(x,0) = \begin{cases} 1 & x \le 0 \\ 1-x & 0 \le x \le 1 \\ 0 & x \ge 1 \end{cases}$

then chars are



What sets the slope of the shock, after it forms? Answer: the "Rankine – Hugoniot condition", which expresses assertion that cons law holds "weakly" across the shock.

Specifically: if shock is at $x = s(t)$ then RH cond says

$$\frac{[F(u)]}{[u]} = \frac{ds}{dt}$$

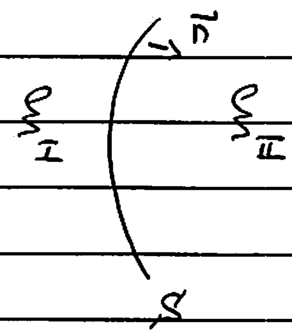
where $[u]$ = jump in u.

Explain: a weak soln should satisfy

$$\iint u\,\psi_t + F(u)\,\psi_x = 0$$

for any compactly sptd $\psi = \psi(x,t)$. Equivalently: The vector field $[u, F(u)]$ is "weakly divergence-free" in (t,x) space.

Let's do this in $\mathbb{R}^n$, since the issue arises in other settings too. Suppose a vector field ξ is piecewise smooth in $\mathbb{R}^n$ but discontinuous across a surface S. When is it "weakly divergence free"? Ans:



ξ is wkly div-free if

a) $\begin{cases} \operatorname{div} \xi_I = 0 & \text{on one side} \\ \operatorname{div} \xi_{II} = 0 & \text{on other side} \end{cases}$

and

b) $\vec{\xi}_I \cdot \vec{n} = \vec{\xi}_{II} \cdot \vec{n}$ at S

Proof: "wkly div-free" means $\int \langle \xi, \nabla\psi \rangle = 0$ for all ψ with cpt spt. Now, with $\vec{n}$ as shown in figure,

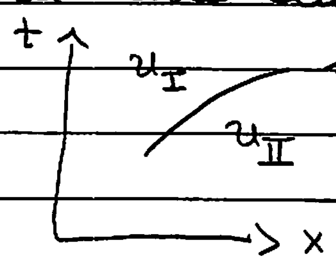
$$\int_I (\operatorname{div}\xi)\,\psi = \int_S (\xi \cdot n)\,\psi - \int_I \langle \xi, \nabla\psi \rangle$$

$$\int_{II} (\operatorname{div}\xi)\,\psi = -\int_S (\xi \cdot n)\,\psi - \int_{II} \langle \xi, \nabla\psi \rangle$$

so $$\int_{I} \langle \xi, \nabla\psi\rangle + \int_{II} \langle \xi, \nabla\psi\rangle = -\int_{I} (\operatorname{div} \xi_{I})\, \psi - \int_{II} (\operatorname{div} \xi_{II})\, \psi$$
$$+ \int_{S} (\xi_{I}\cdot n - \xi_{II}\cdot n)\, \psi$$

LHS vanishes for all ψ $\iff$ $\operatorname{div} \xi_{I} = 0$, $\operatorname{div} \xi_{II} = 0$ and $(\xi_{II} - \xi_{I})\cdot n = 0$ on S

Apply this to the cons law:



shock is at $x = s(t)$

$u_t + F(u)_x = 0 \implies$ use $\xi = (F(u), u)$ in $\mathbb{R}^2 = (x,t)$.
$\vec{n} = (1, -\dot{s}) / (1+\dot{s}^2)^{1/2}$

Wk soln must have

$$F(u_{I}) - \dot{s}\, u_{I} = F(u_{II}) - \dot{s}\, u_{II}$$

$$\iff F(u_{I}) - F(u_{II}) = \dot{s}\,(u_{I} - u_{II})$$

$$\iff \dot{s} = \frac{[F(u)]}{[u]}$$

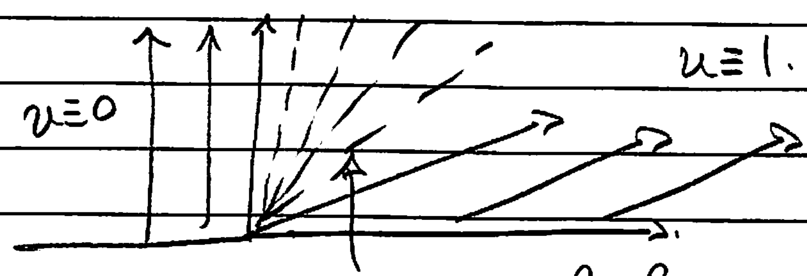
But we're not done, because <u>weak solns</u>

are not unique!

Example: for Burgers' eqn $u_t + \frac{1}{2}(u^2)_x = 0$ with piecewise constant initial data

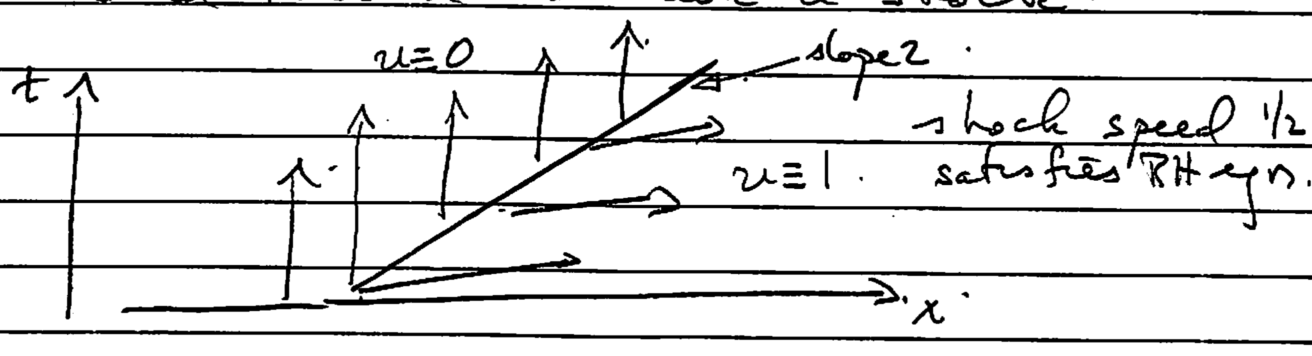
$$u(x,0) = \begin{cases} 0 & x<0 \\ 1 & x>0 \end{cases}$$

we can take u to be given by a "fan"



$u = x/t$ in the fan

or we could take u to have a shock.



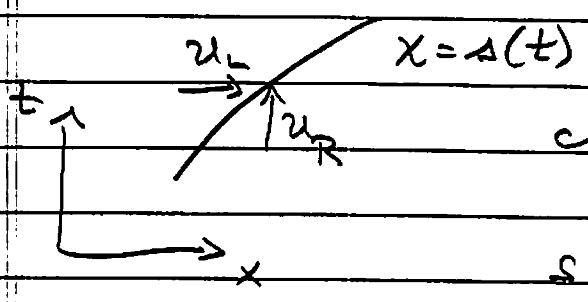
Evidently we must specify which shocks are admissible. For problems from continuum mechanics, there is usually some "viscosity" that was ignored in formulating the cons law's in present setting. This means we expect our $u(x,t)$ to be $\lim_{\varepsilon\to 0} u_\varepsilon(x,t)$ where

$$\partial_t u_\varepsilon + \partial_x F(u_\varepsilon) = \varepsilon\, \partial_{xx} u_\varepsilon .$$

This seems at first hard to use, but it has the following very concrete consequence:

$\boxed{\text{No characteristics should originate at a shock}}$

Explain what this means:

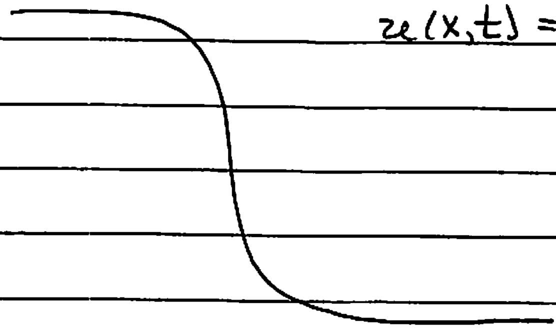


char on left has dirn $(F'(u_L),1)$
" " right " " $(F'(u_R),1)$
shock has dirn $(s'(t),1)$

so picture requires

$$F'(u_L) > \frac{ds}{dt} > F'(u_R)$$

Explanation why it should hold: we expect that for $\varepsilon > 0$ the shock is smoothed to look (locally) like a travelling wave



$$u(x,t) = U\left(\frac{x-\sigma t}{\varepsilon}\right)$$

Then "shock profile" $U(\xi)$ should solve

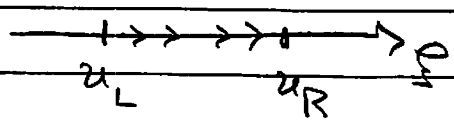
$$-\sigma U' + (F(U))' = U'' \qquad -\infty < \xi < \infty$$

with $U \to u_L, u_R$ as $\xi \to -\infty$ and $+\infty$, and $U' \to 0$ at $\pm\infty$. If such U exists then integrating gives

$$-\sigma U + F(U) = U' + c$$

Evidently $c = -\sigma U_L + F(U_L) = -\sigma U_R + F(U_R)$ (since $U' \to 0$ at $\pm\infty$) which forces $\sigma = [F(U)]/[U]$. We have recovered the RH cond.

But more: the ODE $U' = F(U) - \sigma U - c$ has u_L + u_R as crit pts, and a soln that goes from one to the other



So the linearization should be <u>unstable</u> at u_L

and stable at u_R. This gives

$$F'(u_L) - \sigma \geq 0 \quad , \quad F'(u_R) - \sigma \leq 0$$

Aside from difference btwn strict + nonstrict ineq, this is the same as our condition for admissibility.

For Burgers' eqn we can also see the effect of viscosity directly, by using the "Hopf-Cole transformation". Eqn is

$$u_t + u u_x = \varepsilon u_{xx} \qquad t > 0$$
$$u = g(x) \qquad \text{at } t = 0$$

step 1: integrate in x to get

$$w_t + \tfrac{1}{2} w_x^2 = \varepsilon w_{xx} \qquad t = 0$$
$$w = h(x) \qquad t = 0$$

where $w_x = u$, $h_x = g$.

step 2: reduce to linear heat eqn by ansatz $v = \varphi(w)$. To choose φ, note that if $v = \varphi(w)$ then

$$v_t = \varphi'(w)\, w_t \qquad v_x = \varphi'(w)\, w_x$$

$$v_{xx} = \varphi''(w)\, w_x^2 + \varphi'(w) w_{xx}$$

So $$v_t - \varepsilon v_{xx} = \varphi'(w)\, w_t - \varepsilon \varphi''(w) w_x^2 - \varepsilon \varphi'(w)\, w_{xx}$$

$$= \varphi'(w)\left[\cancel{\varepsilon w_{xx}} - \tfrac{1}{2} w_x^2\right] - \varepsilon\varphi''(w)\, w_x^2 \\ \cancel{- \varepsilon \varphi'(w)\, w_{xx}}$$

For RHS to vanish we should choose φ s.t.

$$\varepsilon \varphi'' + \tfrac{1}{2}\varphi' = 0$$

$$\Rightarrow \text{ use } \varphi(\xi) = e^{-\xi/2\varepsilon} \quad !$$

Thus: $v = e^{-w/2\varepsilon}$ solves the linear heat eqn $v_t - \varepsilon v_{xx} = 0$, with initial data $v(x,0) = e^{-h/2\varepsilon}$.

Net result: $u = \frac{\partial}{\partial x}\left[-2\varepsilon \log v(x,t)\right]$ where

$$v(x,t) = \frac{1}{2\sqrt{\pi\varepsilon t}} \int e^{-(x-y)^2/4\varepsilon t}\, e^{-h(y)/2\varepsilon}\, dy$$

As $\varepsilon \to 0$, the integral is sharply concentrated at the pt y st

$$\frac{(x-y)^2}{4t} + \frac{h(y)}{2} = \min(\text{w r to } y)$$

Use of Laplace asymptotics permits recovery of the "Lax-Oleinik solution formula" specialized to Burgers' eqn: $u = w_x$ where

$$w(x,t) = \min_{y \in \mathbb{R}} \left[\frac{|x-y|^2}{2t} + h(y) \right]$$

(See Evans §4.5.2 for details of this appl'n of Laplace asymptotics.)