PDE - Lecture 1, 9/3/2013

See syllabus for prereqs, semester plan, etc.

While we will discuss many explicit soln formulas, our interest is not so much in being able to "write down" solns. Rather it's in understanding properties of solns, and methods for finding or analyzing solutions. Also in understanding various classes of eqns - where they're from, how solns behave, + how different they can be from one another.

Many books start with 1st order eqns + the method of characteristics. We'll discuss these topics later, starting instead with diffusion eqns such as $u_t = \Delta u$

Motivation 1: heat conduction (see eg Kevorkian §1.1 or Guenther + Lee §1.2)

$$e(x,t) = \text{Thermal energy per unit vol.}$$
$$\vec{q}(x,t) = \text{heat flux vector}$$

$$\frac{d}{dt}\int_V e\,dx = -\int_{\partial V} q\cdot\vec{n}\,ds + \int_V f\,dx$$

expresses cons of energy, where V = any region,

f = energy supplied by internal sources

(note that if q points outward then $\vec{q} \cdot \vec{n} > 0$)

Assuming suff smoothness, + using divergence thm, this reln (for all sets V) $\Leftrightarrow$

$$e_t = -\operatorname{div} q + f.$$

If we assume

$$e = cu \qquad u(x,t) = \text{temperature}$$

$$q = -k \nabla u \qquad \text{"Fick's law"}, \ k > 0$$

Then we get

$$c u_t = \operatorname{div}(k \nabla u) + f.$$

Note that

- eqn is $c u_t = k \Delta u + f$ <u>only</u> if k is indep of x.

- typical bdry conds are
  "Dirichlet" (fix u at bdry of object)
  <u>or</u>
  "Neumann" (fix $q \cdot \vec{n} = -k \frac{\partial u}{\partial n}$ at bdry)

- one easily gets more nonlinear eqns by taking dependence of e or q to be more nonlinear
- a simple model of chemical reaction is

$$c u_t = k \Delta u + f(u) \quad , \text{ where eg } f(u) = u^2 \text{ or } f(u) = e^u$$

Motivation 2: probability

It's easiest to focus on spatially-discrete random walks (which lead to finite-difference approxns of heat eqn or related pde). Focus for simplicity on 1D problems

Most basic example: random walker on 1D lattice of size Δx $\longleftrightarrow$. Walker flips coin at $j\Delta x$

times $n\Delta t$ + goes left or right with equal prob.

Let

$$u(j\Delta x, n\Delta t) = \text{prob of being at node } j\Delta x \text{ at time } n\Delta t$$

Then

$$u(j\Delta x, (n+1)\Delta t) = \tfrac{1}{2} u((j-1)\Delta x, n\Delta t) + \tfrac{1}{2} u((j+1)\Delta x, n\Delta t).$$

So

$$\frac{u(j\Delta x,(n+1)\Delta t) - u(j\Delta x, n\Delta t)}{\Delta t}$$

$$= \frac{(\Delta x)^2}{2\Delta t}\left[\frac{u((j+1)\Delta x, n\Delta t) + u((j-1)\Delta x, n\Delta t) - 2u(j\Delta x, n\Delta t)}{(\Delta x)^2}\right]$$

which if $\frac{(\Delta x)^2}{2\Delta t} = 1$ is a finite-difference discretization of $u_t = u_{xx}$. Initial cond = initial prob density.

Related examples:

a) For same random walker, let $Z(n\Delta t)$ be position at time $n\Delta t$ and consider the "expected final-time reward" at time $T = N\Delta t$:

$$v(j\Delta x, n\Delta t) = \mathbb{E}\left[\varphi(Z(N\Delta t))\right],$$

given that walker is at locn $j\Delta x$ at time $n\Delta t$, $n < N$.

It solves

$$v(j\Delta x, n\Delta t) = \tfrac{1}{2} v((j+1)\Delta x, (n+1)\Delta t) + \tfrac{1}{2} v((j-1)\Delta x, (n+1)\Delta t).$$

since each time step is independent, + walker is at either $(j+1)\Delta x$ or $(j-1)\Delta x$ at next step if he starts at $j\Delta x$. Manipulation as in "most basic example"

$$\frac{v(j\Delta x, n\Delta t) - v(j\Delta x, (n+1)\Delta t)}{\Delta t}$$

$$= \frac{(\Delta x)^2}{2\Delta t}\left[\frac{v((j+1)\Delta x, (n+1)\Delta t) + v((j-1)\Delta x, (n+1)\Delta t) - 2v(j\Delta x, (n+1)\Delta t)}{(\Delta x)^2}\right].$$

which if $\frac{(\Delta x)^2}{2\Delta t} = 1$ is a finite difference version of "heat eqn backward in time"

$$v_t + v_{xx} = 0 \quad \text{for} \quad t < T$$

This is to be solved with a final-time condn

$$v(j\Delta x, N\Delta t) = \varphi(j\Delta x)$$

(or, in continuum limit, $v(x,T) = \varphi(x)$).

Our "most basic example" is the "Forward Kolmogorov eqn" for our random walk (describing evolution of its prob density); our "related example (a)" is the assoc "backward Kolmogorov eqn" (describing expected reward at time T). The fact that forward + backward eqns are related by $t \to -t$ is due to the simplicity of this example; in general the spatial operators are adjoints of one another!

(b) what if our random walker is restricted to the unit interval $0 < x < 1$, in sense that he "dies" if he reaches $x = 0$ or $x = 1$? Now possible posns are $x_j = j/N$, $1 \le j \le N-1$ and $\Delta x = 1/N$. Analogue of our "most basic example" is

$u(j\Delta x, n\Delta t)$ = prob of being still alive at time $n\Delta t$, & being located at node $j\Delta x$

Previous args still apply, giving (if $\frac{(\Delta x)^2}{2\Delta t} = 1$) finite-difference approxn to $u_t = u_{xx}$, but now we have the bdry condition

$$u(t,0) = 0$$
$$u(t,1) = 0$$

as well as the initial condn

$$u(0,x) = \text{initial prob density}$$

Motivation 3: Reaction-diffusion eqns. We already mentioned that an eqn like $u_t - \Delta u = f(u)$

can model chemical reaction, when $f(u)$ = temp-dep source of heat. But in light of our probabilistic discn,

$$u_t = D\Delta u + f(u)$$

can also model a combination of deterministic evolution ($u_t = f(u)$) with diffusion ($u_t = D\Delta u$). For example, a simple model of population growth might be

$$u_t = D\Delta u + c_0 u(1 - c_1 u).$$

("logistic growth model, with diffusion") where now

u = population density
D = diffusion constant
c_0 = growth rate (ignoring diffn) when density is low (so competition is insignificant)
c_1 captures effect of competition on growth rate.

The preceding motivations show that both bounded domains and whole-space problems are of interest. We'll focus on bounded domains first, since they are in some ways simpler.

<u>Uniqueness of classical solutions</u>, <u>by "energy method"</u>

Consider eqn $u_t - \Delta u = 0$ in Ω ($\Omega \subset \mathbb{R}^n$ bdd)

$$u = u_0(x) \quad \text{at } t=0$$

$$u = 0 \quad \text{at } \partial\Omega$$

Multiplying eqn by u + integ by pts gives

$$\frac{1}{2}\frac{d}{dt}\int_\Omega u^2 + \int_\Omega |\nabla u|^2 = 0$$

(with no bdry term, using the bc). This already shows

$$\frac{d}{dt}\int_\Omega u^2 \leq 0$$

which gives uniqueness, since the problem is linear (so diff of two solns has initial data 0).

But we can extract more info by using Poincaré's inequality, which says that if $u|_{\partial\Omega} = 0$ then

$$\int_\Omega u^2\,dx \leq C_\Omega \int_\Omega |\nabla u|^2\,dx$$

(In fact: optimal value of C_Ω satisfies

$$\frac{1}{C_\Omega} = \min_{\substack{u=0 \\ \text{at } \partial\Omega}} \frac{\int_\Omega |\nabla u|^2}{\int_\Omega u^2} = \text{1st eigenvalue of Laplacian with Dir bc ;}$$

we'll return to this later, but it's elementary for $\Omega = [0,1]$ using the Fourier sine series as a basis, cf Guenther + Lee chaps 3+4.)

Using Poincaré's ineq:

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |u|^2 = -\int_\Omega |\nabla u|^2.$$

$$\leq -\frac{1}{C_\Omega}\int_\Omega |u|^2$$

which shows that for any initial cond $u_0(x)$, $\int_\Omega |u|^2$ decays to 0 exponentially fast as $t \to \infty$.

(Warning: the situation is different if we use the Neumann bc $\partial u/\partial n = 0$ at $\partial\Omega$; see HW 1.)

Preceding arg't also gives uniqueness for problem with a source term + inhomog bdry data.

$$u_t - \Delta u = f(x,t) \qquad \text{in } \Omega \times (0,\infty)$$
$$u = g(x,t) \qquad \text{for } x \in \partial\Omega,\ t>0$$
$$u = u_0(x) \qquad \text{at } t=0$$

using linearity (difference of two solns solves the homog problem). If $f + g$ are indep of t we can say more: for

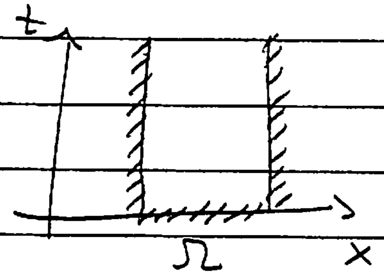
$$u_t - \Delta u = f(x) \qquad \text{in } \Omega$$
$$u = g(x) \qquad \text{at } \partial\Omega$$
$$u = u_0(x) \qquad \text{at } t=0$$

soln decays exponentially to that of stationary problem $-\Delta u_\infty = f$ in Ω, $u_\infty = g$ at $\partial\Omega$ (just consider $u - u_\infty$: its L^2 norm $\to 0$ exp fast as $t \to \infty$, by result proved above).

(Food for thought: what is analogue of the preceding in case of Neumann data, where "steady-state" problem $-\Delta u = f$ in Ω, $\partial u/\partial\nu = g$ at $\partial\Omega$ has either no soln [if $-\int_\Omega f\,dx \neq \int_{\partial\Omega} g\,ds$] or nonunique soln [if $-\int_\Omega f\,dx = \int_{\partial\Omega} g$]?)

Uniqueness of solns, by "max principle". Energy method is great when it works, but "max principle" is more powerful since it applies to a broader class of (2nd order) eqns.

Max prin (weak form, for linear heat eqn): if $u_t - \Delta u = 0$ in $\Omega \times (0,T)$ where Ω is a bdd domain, then max(u) + min(u) are achieved either at the initial time or at the spatial bdry



Pf: As a first pass let's show that, if $u_t - \Delta u < 0$ (strict ≠) then max is achieved at initial time or spatial bdry. In fact (assuming u is C^2)

$$\text{interior max} \Rightarrow u_t = 0,\ \nabla u = 0,\ \Delta u \le 0$$
$$\text{final time max} \Rightarrow u_t \ge 0,\ \nabla u = 0,\ \Delta u \le 0$$

both of which contradict $u_t - \Delta u < 0$.

Second pass: if we only know $u_t - \Delta u \le 0$, then for $\varepsilon > 0$ consider $u_\varepsilon(x,t) = u(x,t) - \varepsilon t$. It has $\partial_t u_\varepsilon - \Delta u_\varepsilon < 0$, so by "1st pass" result

$$\max_{\substack{x \in \Omega \\ 0 \le t \le T}} (u - \varepsilon t) \le \max_{\substack{\text{initial+} \\ \text{spatial bdry}}} (u - \varepsilon t)$$

Now let $\varepsilon \to 0$.

For assertion abt $\min(u)$, apply preceding result to $-u$, or else argue as above with $\varepsilon < 0$ and $u_t - \Delta u \geq 0$.

Note: Essentially the same args apply to

$$u_t - \sum_{i,j=1}^{n} a_{ij}(x,t) \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum b_i(x,t) \frac{\partial u}{\partial x_i} = 0$$

provided $a_{ij}(x,t)$ is nonneg + symmetric, i.e.

$$\sum a_{ij}(x,t) \xi_i \xi_j \geq 0 \quad \text{for all } x \in \Omega, \ \xi \in \mathbb{R}^n.$$

We get uniqueness for $u_t - \Delta u = 0$ (and more general eqns!) with Dir bc $u|_{\partial\Omega} = g$, by subtracting 2 solns + using max prin stated above.

Uniqueness for Neumann bc requires a little more work. Key observation is that if $u_t - \Delta u < 0$ and $\partial u/\partial \nu < 0$ at $\partial\Omega$ then max u is at initial time (pf is same as "first pass" done above). See HW 1.