PCMI – Kohn – Problems for TA Session 1

These problems can be done in any order (they are independent of one another). Their purpose is to provide some practice with the mechanics of thin sheets, and the mathematical models we use to describe them. For the first TA session you'll need to choose, since there is certainly more here than can be done in a single hour.

(1) As we discussed in Lecture 1, when the boundary conditions are consistent with isometry, a piece of paper will minimize its bending energy among isometries.

(a) Consider a square sheet occupying the rectangle $\Omega = (0,1) \times (0,1) \subset R^2$, deformed so that the image of (x, y) is $(u_1(x), y, u_3(x)) \in R^3$ with $(u_1, u_3) = (0, 0)$ at $x = 0$ and $(u_1, u_3) = (0, 1 - \delta)$ at $x = 1$. What is the condition that this deformation be an isometry? Which u minimizes the bending energy among all isometries? What is the associated shape of the piece of paper?

(b) The von-Karman analogue of this question considers in-plane displacement $w = (w_1(x), 0)$ and out-of-plane displacement $u_3(x)$. The membrane energy vanishes when $\partial_x w_1 + \frac{1}{2}(\partial_x u_3)^2 = 0$, and the bending energy is (proportional to) $\int (\partial_{xx} u_3)^2$. What boundary conditions are appropriate? What are the optimal u_3 and w_1? Is your answer consistent with what you found in (a)?

(2) Everyone knows that a stretched elastic string prefers to be straight. Let's look at why, using both the nonlinear and von Karman perspectives. In this problem we'll be focusing on the 1D analogue of the membrane energy (thus, we'll be *ignoring* the bending energy).

(a) Nonlinear first: suppose the reference (stress-free) state of the string is the interval $(0, 1) \subset R$, and the deformed state is $u(x) = (u_1(x), u_2(x)) \in R^2$ with $u(0) = (0, 0)$ and $u(1) = (A, 0)$ with $A > 1$. Use the nonlinear elastic energy $W(u') = (|u'|^2 - 1)^2$. Show that $u(x) = (Ax, 0)$ minimizes $\int_0^1 W(u')\,dx$ with the given boundary conditions. Suppose now $u_2(x_0) = b \neq 0$ for some $x_0 \in (0, 1)$; show that this induces excess energy of at least $8(A^2 - 1)b^2$. (In other words: show that $\int_0^1 W(u')\,dx - (A^2 - 1)^2 \geq 8(A^2 - 1)b^2$.)

(b) The von-Karman analogue of (a) considers tangential displacement $w_1(x)$ and normal displacement $u_2(x)$, and uses the membrane energy $\int_0^1 |\partial_x w_1 + \frac{1}{2}(\partial_x u_2)^2|^2\,dx$ with boundary conditions $(w_1, u_2) = 0$ at $x = 0$ and $(w_1, u_2) = (A - 1, 0)$ at $x = 1$. Show the membrane energy is minimized by $w_1 = (A - 1)x$, $u_2 = 0$. If $u_2(x_0) = b \neq 0$ for some $x_0 \in (0, 1)$, how much excess membrane energy does this induce in the von-Karman setting?

(3) To gain intuition about the (nonlinear) principal stretches associated with a deformation, let's consider an elastic sheet deformed radially, i.e. a map $\phi : R^2 \to R^2$ of the form $\phi(x_1, x_2) = g(r)\frac{x}{r}$ where $r = |x|$, $g(r) > 0$ and $g' > 0$. Show that for $x \neq 0$ the eigendirections of $D\phi^T D\phi$ are radial and tangential, with eigenvalues $(g')^2$ and $(g/r)^2$ respectively (so the principal stretches are g' and g/r). Check that this is consistent with the behavior of ϕ restricted to rays $x/r = \text{const}$, and with the behavior restricted to circles $r = \text{const}$.

(4) We saw in Lecture 1 that if a line is mapped to R^2 by $\phi(x) = (x + w_1(x), u_3(x))$, the von Karman membrane energy $4|\partial_x w_1 + \frac{1}{2}(\partial_x u_3)^2|^2$ arises as the small-slope, small-strain approximation of $(|\partial_x \phi|^2 - 1)^2$. Use a similar argument to show that if a plane is mapped to R^3 by $\phi(x_1, x_2) = (x_1 + w_1(x_1, x_2), x_2 + w_2(x_1, x_2), u_3(x_1, x_2))$ the small-strain, small-slope approximation of $|D\phi^T D\phi - I|^2$ is $4|e(w) + \frac{1}{2}\nabla u_3 \otimes \nabla u_3|^2$.

(5) To gain intuition about the meaning of taking Poisson's ratio to be zero, consider a 2D linearly elastic rectangle $(0,1) \times (0,1)$ with load $\pm(T, 0)$ on the left and right boundaries and elastic energy $W(e) = \frac{1}{2}\mu|e|^2 + \frac{1}{2}\lambda(\text{tr} e)^2$ with $\mu > 0$ and $\lambda \geq 0$. This corresponds to minimization of

$$\int_{[0,1]^2} W(e(u))\, dx_1\, dx_2 - T \int_0^1 u_1(1, x_2)\, dx_2 + T \int_0^1 u_1(0, x_2)\, dx_2$$

where $e(u) = (Du + Du^T)/2$.

(a) Assume for the moment that the interesting deformations are those for which $e(u)$ is a constant, diagonal matrix with eigenvalues e_1, e_2. Show that if $\lambda = 0$ then energy minimization implies $e_2 = 0$, while if $\lambda > 0$ then energy minimization implies $e_2 < 0$.

(b) Show that the assumption of (a) [that it suffices to consider constant, diagonal strains $e(u)$] is correct.

(6) Let B_1 be the unit ball centered at 0 in R^2, and let $g : \partial B_1 \to S^2$ be a unit-speed curve on the unit sphere in R^3.

(a) Show that the associated "conical deformation" $u(x) = |x| g(x/|x|)$ is an isometry for all $|x| > 0$.

(b) Show that this isometry has infinite bending energy.

(c) Show that by modifying it within a ball of radius h, one gets a deformation with (membrane + h^2 bending) energy of order $h^2 |\log h|$. Do the details of the modification influence the leading-order behavior of the energy?

(7) It's clear that a piece of paper has lots of piecewise-isometric deformations with folds that meet at points. The Miura origami pattern the interesting features that (i) it is periodic, and (ii) the interaction of the folds permits a one-parameter family of isometries. In the reference (flat) plane, the locations of the folds are as in the figure. Realize this pattern using a sheet of paper.