Wrinkling of thin elastic sheets – Lecture 4: The herringbone pattern

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Recall from Lecture 1: *tension-induced* wrinkling and *compression-induced* wrinkling are very different.

We understand from Lecture 2 that for tension-induced wrinkling the relaxed problem is nontrivial; it determines the wrinkled region and the direction of the wrinkles. Lecture 3 explored an example.

For compression-induced wrinkling, the relaxed problem is trivial and it provides no guidance. Today’s lecture presents work with Hoai-Minh Nguyen on an example of this type.

Wrinkling of thin films compressed by thick, compliant substrates:

- deposit film at high temp then cool; or
- deposit on stretched substrate then release;
- film buckles to avoid compression

Commonly seen pattern: herringbone

silicon on pdms

gold on pdms
Herringbone pattern when film has some anisotropy, or for specific release histories. Otherwise a less ordered “labyrinth” pattern.

- Silicon on PDMS
  - Song et al., *J Appl Phys* 103 (2008) 014303

- Gold on PDMS

- Different release histories
Using the von Karman framework, the energy has three terms:

1. **Membrane energy** captures the fact that film’s natural length is larger than that of the substrate:
   \[ \alpha_m h \int \left| e(w) + \frac{1}{2} \nabla u_3 \otimes \nabla u_3 - \eta I \right|^2 \, dx \, dy \]

2. **Bending energy** captures the resistance to bending:
   \[ h^3 \int \left| \nabla \nabla u_3 \right|^2 \, dx \, dy \]

3. **Substrate energy** captures the fact that the substrate acts as a “spring”, tending to keep the film flat:
   \[ \alpha_s \left( \| w \|^2_{H^{1/2}} + \| u_3 \|^2_{H^{1/2}} \right) \]

   where \( \| g \|^2_{H^{1/2}} = \sum \| k \| \hat{g}(k) \|^2 \)

Membrane energy is proportional to \( h \), and bending to \( h^3 \), since substrate energy is proportional to area.
The membrane energy

\[ E_{\text{membrane}} = \alpha_m h \int |e(w) + \frac{1}{2} \nabla u_3 \otimes \nabla u_3 - \eta I|^2 \, dx \, dy \]

where \((w_1, w_2, u_3)\) is the elastic displacement, and \(\eta > 0\) is the misfit (nondimensional but small). Keeping \(\alpha_m\) as a parameter permits us to see when the membrane term is important.

- For membrane term to be small, expect \(|e(w)| \sim \eta\) and \(|\nabla u_3| \sim \sqrt{\eta}\).
- For 1D analogue \(\int |\partial_x w_1 + \frac{1}{2} (\partial_x u_3)^2 - \eta|^2 \, dx\), integrand vanishes eg if wrinkling profile is sinusoidal,
  \[ w_1 = \eta \frac{\lambda}{4} \sin(4x/\lambda), \quad u_3 = \sqrt{\eta} \lambda \cos(2x/\lambda) \]

- Our problem is 2D, with isotropic misfit \(\eta I\);
  membrane term would vanish for piecewise-linear “Miura ori” pattern.

- The herringbone pattern uses sinusoidal wrinkling in two distinct orientations. It does better than the Miura-ori pattern.
The substrate energy

\[ E_{\text{substrate}} = \alpha_s \left( \| w \|_{H^{1/2}}^2 + \| u_3 \|_{H^{1/2}}^2 \right) \]

where \((w_1, w_2, u_3)\) are assumed periodic (on some large scale \(L\)),

\[ \| g \|_{H^{1/2}}^2 = \sum |k| |\hat{g}(k)|^2 \]

and

\[ \alpha_s = \text{substrate stiffness/film stiffness}. \]

- Treat substrate as semi-infinite isotropic elastic halfspace.
- Given surface displacement \((w_1, w_2, u_3)\), solve 3D linear elasticity problem in substrate by separation of variables.
- Substrate energy is the result (modulo constants).
Total energy = membrane + bending + substrate

To permit spatial averaging, we assume periodicity on some (large) scale $L$, and we focus on the energy per unit area:

$$ E_h = \frac{\alpha_m h}{L^2} \int_{[0,L]^2} \left| e(w) + \frac{1}{2} \nabla u_3 \otimes \nabla u_3 - \eta I \right|^2 \, dx \, dy \quad \text{(membrane)} $$

$$ + \frac{h^3}{L^2} \int_{[0,L]^2} \left| \nabla \nabla u_3 \right|^2 \, dx \, dy \quad \text{(bending)} $$

$$ + \frac{\alpha_s}{L^2} \left( \| w \|_{H^1/2}^2 + \| u_3 \|_{H^1/2}^2 \right) \quad \text{(substrate)} $$

where $h$ is the thickness of the film.

- We have already normalized by stiffness of the film, so $\alpha_m, \alpha_s, \eta$ are dimensionless parameters:
  - $\alpha_m$ (order 1) comes from mechanics of bending;
  - $\alpha_s$ (small) is the ratio (substrate stiffness)/(film stiffness);
  - $\eta$ (small, pos) is the misfit.

- Unwrinkled state $(w_1, w_2, u_3) = 0$ has energy $\alpha_m \eta^2 h$. 

Wrinkling – Lecture 4
The energy scaling law

**Theorem**

If $h/L$ and $\eta$ are small enough, the minimum energy satisfies

$$\min E_h \sim \min \{ \alpha_m \eta^2 h, \alpha_s^{2/3} \eta h \};$$

moreover

- the first alternative corresponds to the *unwrinkled state*; it is better when $\alpha_m \eta < \alpha_s^{2/3}$.
- the second alternative is achieved by a *herringbone pattern* using wrinkles with length scale $\alpha_s^{-1/3} h$, whose direction oscillates on a suitable scale (longer but not fully determined).

The smallness conditions are explicit:

$$\alpha_m \alpha_s^{-4/3} \left( \frac{h}{L} \right)^2 \leq 1 \quad \text{and} \quad \eta^2 \leq \alpha_m^{-1} \alpha_s^{2/3}. $$

Perhaps other, less-ordered patterns could also be optimal (e.g. “labyrinths”).

Wrinkling – Lecture 4
The energy scaling law – cont’d

\[ \min E_h \sim \min \{ \alpha_m \eta^2 h, \alpha_s^{2/3} \eta h \}; \]

One consequence: the Miura-ori pattern is not optimal:

- Its scaling law is \( \alpha_m^{1/6} \alpha_s^{5/8} \eta^{17/16} h \).
- If film prefers not to be flat (\( \alpha_m \eta \gg \alpha_s^{2/3} \)) then Miura-ori energy \( \gg \) herringbone energy.

Intuition:

- Bending energy requires folds of Miura-ori pattern to be rounded.
- Where folds intersect this costs significant membrane energy.
- In herringbone pattern the membrane term isn’t identically zero, but it does not contribute at leading order.
Tasks for analysis

Our assertion is:

\[
\min E_h \geq C_1 \min \{\alpha_m \eta^2 h, \alpha_{s}^{2/3} \eta h\} \text{ and } \\
\min E_h \leq C_2 \min \{\alpha_m \eta^2 h, \alpha_{s}^{2/3} \eta h\}
\]

with \( C_1, C_2 \) independent of \( h, \alpha_s, \) and \( \alpha_m \), provided

\[
\alpha_m \alpha_{s}^{-4/3} (h/L)^2 \leq 1 \quad \text{and} \quad \eta^2 \leq \alpha_m^{-1} \alpha_{s}^{2/3}.
\]

Our tasks are thus

- to prove an upper bound, by describing and optimizing the herringbone pattern; and
- to prove a “matching” ansatz-free lower bound.
The upper bound – overview

- Energy of unbuckled state is $\alpha m \eta^2 h$
- Energy of herringbone is $C \alpha_s^{2/3} \eta h$
- So $\min E_h \leq \min \{\alpha m \eta^2 h, C \alpha_s^{2/3} \eta h\}$

$$
E_h = \frac{\alpha_m h}{L^2} \int_{[0,L]^2} \left| e(w) + \frac{1}{2} \nabla u_3 \otimes \nabla u_3 - \eta I \right|^2 \, dx \, dy \\
+ \frac{h^3}{L^2} \int_{[0,L]^2} \left| \nabla \nabla u_3 \right|^2 \, dx \, dy + \frac{\alpha_s}{L^2} \left( \|w\|_{H^{1/2}}^2 + \|u_3\|_{H^{1/2}}^2 \right)
$$

Key features of herringbone:

- membrane term is negligible
- typical slope is $|\nabla u_3| \sim \sqrt{\eta}$
- typical in-plane strain is $|e(w)| \sim \eta$ (smaller!)
- scale of wrinkling set by competition between bending term and $u_3$ part of substrate term
- two types of wrinkling must mix for $e(w)$ to have average 0; but the longer length scale is not fully determined.
More detail on the herringbone pattern

Film wants to expand (isotropically) relative to substrate.
- 1D wrinkling expands only transverse to the wrinkles
- a simple shear expands one diag dirn, compresses the other
- shear combined with wrinkling achieves isotropic expansion

Substrate prohibits large deformation; therefore the film mixes the two shear-combined-with-wrinkling variants. Thus, the herringbone pattern has two length scales:

- The smaller one (the scale of the wrinkling) is set by competition between bending term and substrate energy of $u_3$.
- The larger one (scale of the phase mixture) must be s.t. the substrate energy of $w$ is insignificant. (It is not fully determined.)
Another perspective

- Mixture of two symmetry-related “phases”
- “Phase 1” uses sinusoidal wrinkles perp to $(1, 1)$, superimposed on an in-plane shear.
- “Phase 2” uses wrinkles perp to $(1, -1)$, superimposed on a different shear.

In phase 1: \[ e(w) + \frac{1}{2} \nabla u_3 \otimes \nabla u_3 = \begin{pmatrix} 0 & -\eta \\ -\eta & 0 \end{pmatrix} + \begin{pmatrix} \eta & \eta \\ \eta & \eta \end{pmatrix} = \eta I; \]

In phase 2: \[ e(w) + \frac{1}{2} \nabla u_3 \otimes \nabla u_3 = \begin{pmatrix} 0 & \eta \\ \eta & 0 \end{pmatrix} + \begin{pmatrix} -\eta & -\eta \\ -\eta & -\eta \end{pmatrix} = \eta I; \]

Membrane term vanishes (except in transition layers between the phases)! Since avg in-plane shear is 0, in-plane displacement $w$ can be periodic.

Scale of shear oscillation can be much longer than scale of wrinkling, since $e(w) \sim \eta$ while $u_3 \sim \sqrt{\eta}$, and $\eta \ll 1$.

Scale of shear oscillation must be small enough: substrate energy of shear osc $\leq$ substrate energy of wrinkling.

Scale of shear oscillation must be large enough: membrane energy of transition layers should be $\leq$ other energy assoc with wrinkling.
Phenomenology - review

silicon on pdms
Song et al, J Appl Phys 103 (2008) 014303

gold on pdms
Chen & Hutchinson, Scripta Mat 50 (2004) 797–801

different release histories
No other pattern can do better

Want to show: For any periodic \((w_1, w_2, u_3)\), \(E_h \geq C \min\{\alpha_m \eta^2 h, \alpha_s^{2/3} \eta h\}\).

The proof is surprisingly easy. To simplify notation, take the period to be \(L = 1\). We’ll use only that

\[
\text{membrane term} \geq \alpha_m h \int |\partial_1 w_1 + \frac{1}{2} |\partial_1 u_3|^2 - \eta|^2 \, dx \, dy,
\]

\[
\text{bending term} = h^3 \|\nabla \nabla u_3\|_{L^2}^2,
\]

and \(\text{substrate term} \geq \alpha_s \|u_3\|_{H^{1/2}}^2\).

CASE 1, FIRST PASS: If \(\int (\partial_1 u_3)^2 \ll \eta\) then membrane \(\gtrsim \alpha_m \eta^2 h\), since \(\partial_x w_1\) has mean 0.

CASE 2, FIRST PASS: If \(\int (\partial_1 u_3)^2 \gg \eta\) use the interpolation inequality

\[
\|\nabla u_3\|_{L^2} \leq \|\nabla \nabla u_3\|_{L^2}^{1/3} \|u_3\|_{H^{1/2}}^{2/3}
\]

to see that

\[
\text{Bending + substrate terms} = h^3 \|\nabla \nabla u_3\|_{L^2}^2 + \frac{1}{2} \alpha_s \|u_3\|_{H^{1/2}}^2 + \frac{1}{2} \alpha_s \|u_3\|_{H^{1/2}}^2 \\
\gtrsim \left(h^3 \|\nabla \nabla u_3\|_{L^2}^2 \alpha_s^2 \|u_3\|_{H^{1/2}}^4\right)^{1/3} \\
\gtrsim h \alpha_s^{2/3} \|\nabla u_3\|_{L^2}^2 \gtrsim h \alpha_s^{2/3} \eta
\]
Want to show: For any periodic \((w_1, w_2, u_3)\), \(E_h \geq C \min\{\alpha_m \eta^2 h, \alpha_s^{2/3} \eta h\}\).

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\]

to see that

\[
\text{Bending + substrate terms} = h^3 \| \nabla \nabla u_3 \|^2 + \frac{1}{2} \alpha_s \| u_3 \|_{H^{1/2}}^2 + \frac{1}{2} \alpha_s \| u_3 \|_{H^{1/2}}^2
\]

\[
\geq \left( h^3 \| \nabla \nabla u_3 \|^2 \alpha_s^2 \| u_3 \|_{H^{1/2}}^4 \right)^{1/3}
\]

\[
\geq h \alpha_s^{2/3} \| \nabla u_3 \|^2_{L^2} \geq h \alpha_s^{2/3} \eta
\]
Want to show: For any periodic \((w_1, w_2, u_3)\), \(E_h \geq C \min\{\alpha_m \eta^2 h, \alpha_s^{2/3} \eta h\}\).

The proof is surprisingly easy. To simplify notation, take the period to be \(L = 1\). We’ll use only that

\[
\text{membrane term} \geq \alpha_m h \int \left| \partial_1 w_1 + \frac{1}{2} |\partial_1 u_3|^2 - \eta \right|^2 \, dx \, dy,
\]

\[
\text{bending term} = h^3 \|\nabla \nabla u_3\|_{L^2}^2, \quad \text{and} \quad \text{substrate term} \geq \alpha_s \|u_3\|_{H^{1/2}}^2.
\]

**CASE 1, FIRST PASS:** If \(\int (\partial_1 u_3)^2 \ll \eta\) then membrane \(\gtrsim \alpha_m \eta^2 h\), since \(\partial_x w_1\) has mean 0.

**CASE 2, FIRST PASS:** If \(\int (\partial_1 u_3)^2 \gtrsim \eta\) use the interpolation inequality

\[
\|\nabla u_3\|_{L^2} \lesssim \|\nabla \nabla u_3\|_{L^2}^{1/3} \|u_3\|_{H^{1/2}}^{2/3}
\]

to see that

\[
\text{Bending + substrate terms} = h^3 \|\nabla \nabla u_3\|_{L^2}^2 + \frac{1}{2} \alpha_s \|u_3\|_{H^{1/2}}^2 + \frac{1}{2} \alpha_s \|u_3\|_{H^{1/2}}^2
\]

\[
\gtrsim \left( h^3 \|\nabla \nabla u_3\|_{L^2}^2 \alpha_s^2 \|u_3\|_{H^{1/2}}^4 \right)^{1/3}
\]

\[
\gtrsim h \alpha_s^{2/3} \|\nabla u_3\|_{L^2}^2 \gtrsim h \alpha_s^{2/3} \eta
\]
CASE 1, SECOND PASS: Suppose $\int \frac{1}{2}(\partial_1 u_3)^2 \leq \frac{1}{2} \eta$. Use the inequality

$$\int \partial_1 w_1 + \frac{1}{2}(\partial_1 u_3)^2 - \eta \leq \left( \int |\partial_1 w_1 + \frac{1}{2}(\partial_1 u_3)^2 - \eta|^2 \right)^{1/2}$$

(recall that we took $L = 1$ to simplify the notation). Since $w_1$ is periodic,

$$\text{LHS} = \int \frac{1}{2}(\partial_1 u_3)^2 - \eta \geq \frac{1}{2} \eta.$$  

So

$$h_{\alpha m} \int |\partial_1 w_1 + \frac{1}{2}(\partial_1 u_3)^2 - \eta|^2 \geq h_{\alpha m} (\frac{1}{2} \eta)^2$$
CASE 2, SECOND PASS: Suppose \( \int \frac{1}{2} |\partial_1 u_3|^2 \geq \frac{1}{2} \eta \). Then evidently

\[
\int |\nabla u_3|^2 \geq \eta.
\]

Our first pass argument combined the inequality \( \frac{a+b+c}{3} \geq (abc)^{1/3} \) with an interpolation inequality, which can be written

\[
\int |\nabla u|^2 \leq \left( \int |\nabla \nabla u|^2 \right)^{1/3} \left( \int |\nabla^{1/2} u|^2 \right)^{2/3}
\]

using the suggestive notation \( \sum |k| |\hat{u}(k)|^2 = \int |\nabla^{1/2} u|^2 \). Proof of this ineq is easy in Fourier space:

\[
\sum |k|^2 |\hat{u}(k)|^2 = \sum |k|^{4/3} |\hat{u}(k)|^{2/3} \cdot |k|^{2/3} |\hat{u}(k)|^{4/3} \leq \left( \sum |k|^4 |\hat{u}(k)|^2 \right)^{1/3} \left( \sum |k| |\hat{u}(k)|^2 \right)^{2/3}
\]
Stepping back

Main accomplishment: scaling law of the minimum energy, based on
- upper bound, corresponding to the herringbone pattern, and
- lower bound, using little more than interpolation.
- Key point: they agree (up to a factor indep of $h$, $\eta$, and $\alpha_s$).

Open question: what about those labyrinth patterns?
- Why are they seen in some numerical and physical experiments (but not in others)? Do they achieve the same scaling law, or are they higher-energy local minima?
A closely related problem

What if the film can relieve the misfit by blistering?

FeNi on salt
(from Gioia & Ortiz, 1997)

Very different from perfectly-bonded case, since substrate feels only in-plane displacement of bonded region.

Recent joint work with Jacob Bedrossian (CPAM in press, and preprint at arxiv): there is a regime where a lattice-like blistering pattern is energetically preferred over a few large blisters (if the area fraction of blistering is fixed).

