Wrinkling of Thin Elastic Sheets – Lecture 1

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Goals for Lecture 1

Getting started:

- A first look at some phenomena involving
  - paper
  - tension-induced wrinkling
  - compressive wrinkling
- What kind of math underlies such problems?
  - nonconvex variational problems with a small regularization
- The elastic energy of a thin sheet
  - a fully nonlinear model
  - a small-strain, small-slope (von Karman) model

Paper, deformed smoothly

Paper is (almost) inextensible. So we can describe deformations of a flat piece of paper using

$$g : \mathbb{R}^2 \to \mathbb{R}^3 \quad \text{such that } (Dg)^T Dg = I.$$

Explanation:

$$|Dg \cdot v|^2 = 1 \Leftrightarrow \langle v, (Dg)^T Dg\, v\rangle = 1;$$

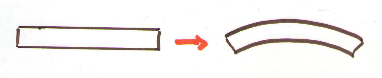
and this must hold for all unit vectors v in $\mathbb{R}^2$.

Facts from geometry:

- A surface in $\mathbb{R}^3$ has 2 principal curvatures κ_1, κ_2 (eigenvalues of quadratic form associated to quadratic approxn).
- For image of a smooth isometry, $\kappa_1 \kappa_2 = 0$ pointwise.
- Image of a smooth isometry is a developable surface.

Paper, deformed smoothly

Essential mechanics of paper: it resists bending.



If midline is isometric but image is curved, then lines above/below middle are stretched/shrunk. If thickness is h and curvature is κ, then

$$\text{unhappiness} = \int_0^L \int_{-h/2}^{h/2} z^2 \kappa^2 \, dz \, dx = ch^3 \int_0^L \kappa^2 \, dx.$$

Basic model of paper:

$$\min \int_\Omega \kappa_1^2 + \kappa_2^2 \, dA$$

where $\Omega \subset \mathbb{R}^2$, $g : \Omega \to \mathbb{R}^3$ is an isometry, and κ_1, κ_2 are the principal curvatures of the image.

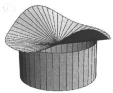
Not necessarily easy to solve (what is the shape of a Mobius band?).

Boundary conditions matter; sometimes gravity matters too.

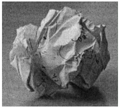
Paper, with singularities

Actually: smooth isometries are not sufficient. Depending on loads and bdry conds, we easily see

- formation of point singularities ("d-cones")



- formation of line singularities ("crumpling")



Why? Because there is no isometry with finite bending energy.

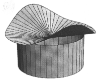
Heuristic calculation: consider a "perfectly conical d-cone": use $\Omega = \{r < 1\}$ and $g(r, \theta) = r\varphi(\theta)$ where $\varphi(\theta)$ is a curve on S^2 with the right length (2π). Then

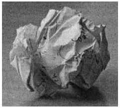
$$\text{curvature at radius } r \sim \frac{1}{r}$$

so

$$\int \kappa^2 \, dA \sim \int_0^1 \frac{1}{r^2} r dr \quad \text{is divergent.}$$

Paper, with singularities





How to model d-cones and crumpling? Main idea:

$$E_h = \min \int_\Omega |(Dg)^T Dg - I|^2 \, dA + h^2 \int_\Omega \kappa_1^2 + \kappa_2^2 \, dA$$

First term ("membrane energy") prefers isometry. It is nonconvex (for paper: many smooth minimizers).

Second term ("bending energy") resists bending, but has h^2 in front.

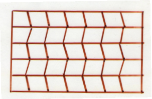
If bc permit isometry with curvature in L^2 then $E_h \sim h^2$ as $h \to 0$.

For d-cone and crumpling, $E_h \gg h^2$ as $h \to 0$. (How does E_h behave as $h \to 0$? We have conjectures, but few theorems.)

Digression: the Miura folding pattern

Problem: NYC subway map is difficult to fold "correctly."

Solution: The Miura map is easy to fold "correctly."

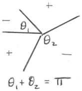


Is there math here?

$$\text{folding paper flat} \Leftrightarrow g : \mathbb{R}^2 \to \mathbb{R}^2 \text{ such that } Dg \in O(2).$$

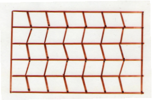
So g is locally a rotation (preserving or reversing orientation).

When 4 creases meet, opposite angles must add to π (to permit folding flat).



- For arbitary "creases" (violating angle condition) there is no way to fold paper flat using them.
- For rectangular creases there are many ways. For Miura pattern there is essentially one way.

Digression: the Miura folding pattern



Actually, there's a lot of math here:

- To have finite energy, the folds must be smoothed. To minimize energy

$$E_h = \min \int |(Dg)^T Dg - I|^2 \, dA + h^2 \int \kappa_1^2 + \kappa_2^2 \, dA$$

  scale of smoothing should depend on dist from fold-crossings (Lobkovsky-Witten, Venkataramani, Conti-Maggi). Best possible (if length scale of folds is of order 1) is $E_h \sim h^{5/3}$.

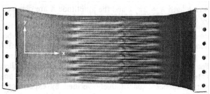
- Scaling so that slopes remain fixed and amplitude $\to 0$ gives an (asymptotically) piecewise-isometric approximation of uniform compression with $E_h \sim h^{5/3-\delta}$ for any $\delta > 0$. (Is this the best one can do? Is it why crumpling induces folds?)

Tension-induced wrinkling

Paper is special, because it's almost inextensible. In more elastic thin sheets, we often see wrinkling. Useful to distinguish between tension-induced and compressive wrinkling. Some tension-induced examples:

- hanging drapes (Vandeparre et al, PRL 2011)

- stretched sheets (Cerda & Mahadevan, PRL 2003)



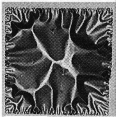
- water drop on floating sheet (Huang et al, Science 2007)

Common features: membrane effects induce uniaxial tension. Wrinkles serve to avoid compression. Wrinkling direction is known. Scale of wrinkling may depend on location. As $h \to 0$, scale of wrinkling $\to 0$.

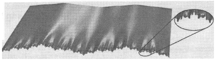
Compressive wrinkling

Some examples of compressive wrinkling:

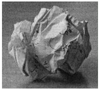
- Compression due to thermal mismatch: a thin film bonded to a too-short bdry (Lai et al, J Power Sources 2010)



- Metric-driven wrinkling, proposed as a model for growth-induced wrinkling in leaves and flowers (numerics, Audoly & Boudaoud, PRL 2003)



- Crumpling is not so different!



More difficult than tension-induced wrinkling:

- Should we expect wrinkles or folds?
- Direction of wrinkles/folds not clear in advance.
- Greater multiplicity of low-energy structures.

Patterns vs scaling laws

$$E_h = \min\{\text{membrane energy} + h^2 \text{ bending energy} + \text{term assoc to loading}\}.$$

Setting $h = 0$ leads to infinite bending.

Problem: It is difficult to *describe* a pattern (even a well-organized one, as in the figures), let alone explain why it occurs.

Solution: Focus instead on the *scaling law* of the minimum energy, i.e. find asymptotics of E_h as $h \to 0$. For example, do there exist E_0 and α such that

$$E_0 + C_1 h^\alpha \leq E_h \leq E_0 + C_2 h^\alpha ?$$

- Upper bound can be obtained by guessing form of solution.
- Lower must consider *any* pattern (even those not seen in nature).
- Proving the upper bound involves describing the pattern.
- Proving the lower bound involves understanding what drives it.

The variational perspective

Program: study asymptotics of E_h, for example

$$E_0 + C_1 h^\alpha \leq E_h \leq E_0 + C_2 h^\alpha .$$

- If $\alpha = 2$ then curvature is unif bounded in L^2, hence no microstructure.
- Simulation is also useful. But:
  - increasingly stiff as $h \to 0$
  - hard to explore global min this way
  - simulation shows how a pattern forms, not so much why it forms.
- Bifurcation is also useful. But min energy state lies deep in the bifurcation diagram as $h \to 0$.
- Minimization within an ansatz is widely used. But is the ansatz adequate? Yes, if there's a matching lower bound.

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The variational perspective – cont'd

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$$E_0 + C_1 h^\alpha \leq E_h \leq E_0 + C_2 h^\alpha .$$

- How to approach lower bound? In convex problems, lower bounds come from duality. But our problems are highly nonconvex. No universal method yet, but techniques are beginning to emerge through examples.
- What about the pattern? I'll focus mainly on asymptotics of the energy. Of course I'm also interested in ptwise details of energy-min pattern, but rigorous ptwise results are known only in a few cases.

The variational perspective – cont'd

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Getting started

Now some basic mechanics, laying groundwork for the first TA Session and our variational analysis of wrinkling. Topics:

- a 1D elastic spring (constrained to a line)
- a 1D elastic spring (in 3D, but ignoring bending)
- a 2D elastic sheet (membrane energy, nonlinear version)
- a 2D elastic sheet (membrane energy, von Karman version)
- a 2D elastic sheet (the bending energy)

A 1D elastic spring, constrained to a line

Reference state: $[0, L]$ (Stress-free). Deformation: $u : [0, L] \to \mathbb{R}$ with $u_x > 0$.

Elastic energy: If $x = 0$ is fixed and we pull by force T at RHS,

$$\min_{u(0)=0} \int_0^L W_{1D}(u')\, dx - u(L)T$$

Euler Lagrange eqn expresses force balance:

$$\frac{d}{dx}\left[W'_{1D}(u')\right] = 0, \quad \text{with } W'_{1D}(u') = T \text{ at } x = L.$$

- u' is the "stretch" (string prefers $u' = 1$); $e = u' - 1$ is the "nonlinear strain" (string prefers $e = 0$).
- If we expect $u' \approx 1$, then it is reasonable to take $W_{1D}(\lambda) = c|\lambda - 1|^2$.
- In general: W'_{1D} is the stress (force) assoc to stretch u'; so $W_{1D}(\lambda)$ should be min at $\lambda = 1$, and $W_{1D}(\lambda) \to \infty$ as $\lambda \downarrow 0$ or $\lambda \to \infty$.

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A 1D elastic spring in $\mathbb{R}^3$

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$$\min_{u(0)=0} \int_0^L W_{1D}(|u'|)\, dx - u(L) \cdot T$$

Euler Lagrange eqn still expresses force balance:

$$\frac{d}{dx}\left[W'_{1D}(|u'|) \frac{u'}{|u'|} \right] = 0, \quad \text{with } W'_{1D}(u') \tfrac{u'}{|u'|} = T \text{ at } x = L.$$

- $|u'|$ is the stretch (string prefers $|u'| = 1$); $e = |u'| - 1$ is the strain.
- W'_{1D} is the magnitude of the stress (force) assoc to stretch $|u'|$ (negative for $|u'| < 1$, positive for $|u'| > 1$).
- When viewed as a function of $u' \in \mathbb{R}^3$, $W_{1D}(u')$ is nonconvex (it is min on the circle $|u'| = 1$).
- For small strain, reasonable to take $W_{1D}(|u'|) = c(|u'| - 1)^2 = ce^2$.
- However: since $|u'|^2 = (1 + e)^2 \approx 1 + 2e$, equally reasonable to take $W_{1D} = \frac{c}{4}(|u'|^2 - 1)^2$.

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The membrane energy of a 2D sheet

Reference state: $\Omega \subset \mathbb{R}^2$ (Stress-free). Deformation: $g : \Omega \to \mathbb{R}^3$.

Polar decomposition: For any $x \in \Omega$, the lin approx $Dg(x)$ can be expressed as a product: $Dg(x) = Q \cdot (Dg^T Dg)^{1/2}$, where Q is an isometry of $\mathbb{R}^2$ to $\mathbb{R}^3$.

Principal stretches λ_1, λ_2 are the eigenvalues of $(Dg^T Dg)^{1/2}$. Principal strains are $e_i = \lambda_i - 1$. Principal directions are eigenvectors of $Dg^T Dg$.

For an isotropic membrane, the membrane energy W_m is a symmetric function of λ_1 and λ_2. If it's quadratic in e_1 and e_2, then it must be

$$W_m(Dg) = c_1(e_1 + e_2)^2 + c_2(e_1^2 + e_2^2).$$

To keep things simple, I'll often take $c_1 = 0$ ("Poisson's ratio zero").

My favorite model is slightly different:

$$W_m(Dg) = |Dg^T Dg - I|^2.$$

Remembering that

$$Dg^T Dg \sim \begin{pmatrix} \lambda_1^2 & 0 \\ 0 & \lambda_2^2 \end{pmatrix} = \begin{pmatrix} (1+e_1)^2 & 0 \\ 0 & (1+e_2)^2 \end{pmatrix},$$

for small strain this is equivalent (at leading order) to $4(e_1^2 + e_2^2)$.

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The von Karman viewpoint – warmup

Preceding discussion assumed small strain but allowed arbitrarily large change of orientation. The von Karman viewpoint is different because it assumes the sheet is nearly flat.

To see the main idea: consider a horizontal string mapped into $\mathbb{R}^2$, with horizontal displacement w and transverse displacement u. This amounts to considering considering $g : [0, L] \to \mathbb{R}^2$:

$$g(x) = (x + w(x), u(x)).$$

We have

$$|g'| = [(1 + w')^2 + u'^2]^{1/2} \approx 1 + w' + \tfrac{1}{2}u'^2$$

if w' and u' are both small. So the strain is approx

$$e = w' + \tfrac{1}{2}u'^2$$

and our typical quadratic energy becomes

$$W = c|w' + \tfrac{1}{2}u'^2|^2.$$

The von Karman membrane energy of a 2D sheet

The 2D case is similar to the warmup. Reference state is now $\Omega \subset \mathbb{R}^2 \times \{0\}$.

In describing the deformation we distinguish between the *in-plane displacement* $w : \Omega \to \mathbb{R}^2$ and the *out-of-plane displacement* $u_3 : \Omega \to \mathbb{R}$.

Assuming isotropy and Poisson's ratio zero, von Karman membrane energy is

$$W_m = c|e(w) + \tfrac{1}{2}\nabla u_3 \otimes \nabla u_3|^2$$

where $e(w)$ is the "linear elastic strain"

$$e(w) = \frac{\nabla w + (\nabla w)^T}{2} = \begin{bmatrix} \partial_1 w_1 & \frac{\partial_1 w_2 + \partial_2 w_1}{2} \\ \frac{\partial_1 w_2 + \partial_2 w_1}{2} & \partial_2 w_2 \end{bmatrix}$$

and $\nabla u_3 \otimes \nabla u_3$ denotes the rank-one matrix

$$\nabla u_3 \otimes \nabla u_3 = \begin{bmatrix} (\partial_1 u_3)^2 & \partial_1 u_3\, \partial_2 u_3 \\ \partial_1 u_3\, \partial_2 u_3 & (\partial_2 u_3)^2 \end{bmatrix}$$

Notice that our 1D warmup is just the special case where $w_2 = 0$ and w_1, u_3 depend only on x_1.

The correspondence between the 2D nonlinear viewpoint and the 2D von Karman viewpoint is entirely parallel to our 1D warmup.

The bending energy

For a 1D string in $\mathbb{R}^2$, if $W_{1D} = |e|^2$, then the bending energy per unit thickness is

$$\tfrac{1}{12} h^2 \int \kappa^2 \, dx$$

where h is the thickness. Why $\frac{1}{12}h^2$? Because at distance z from the midline, $|e|^2 \approx |\kappa z|^2$, and

$$\int_{-h/2}^{h/2} z^2 \, dz = \tfrac{1}{12} h^3.$$

We drop one power of h, because we want energy per unit thickness.

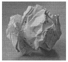
For a 2D sheet modeled using the von Karman framework, the analogous bending energy is

$$\tfrac{1}{12} h^2 \int |\nabla\nabla u_3|^2 \, dx$$

since in the small-slope, small-deformation limit, the principal curvatures are the eigenvalues of $\nabla\nabla u$.

I'll often drop the factor $1/12$ to avoid clutter.

Image credits



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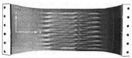

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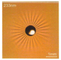

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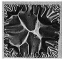

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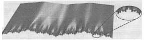

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