The Mathematics of Wrinkles and Folds

Robert V. Kohn
Courant Institute, NYU

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Wrinkles and folds

We’ll be talking about

- inextensible sheets (like paper)
- extensible sheets (rubber or cloth)
- thin crystalline films on substrates
- a little geometry and mechanics
- a lot of calculus of variations
Paper, deformed smoothly

Paper is (almost) inextensible. So we can describe deformations of a flat piece of paper using

\[ g : \mathbb{R}^2 \rightarrow \mathbb{R}^3 \]  such that \((Dg)^T Dg = I.\]

Explanation:

\[ |Dg \cdot v|^2 = 1 \iff \langle v, (Dg)^T Dg \cdot v \rangle = 1; \]

and this must hold for all unit vectors \(v\) in \(\mathbb{R}^2\).

Facts from geometry:

- A surface in \(\mathbb{R}^3\) has 2 principal curvatures \(\kappa_1, \kappa_2\) (eigenvalues of quadratic form associated to quadratic approxn).
- For image of a smooth isometry, \(\kappa_1 \kappa_2 = 0\) pointwise.
- Image of a smooth isometry is a developable surface.
Essential mechanics of paper: it resists bending.

If midline is isometric but image is curved, then lines above/below middle are stretched/shrunk. If thickness is $h$ and curvature is $\kappa$, then

$$\text{unhappiness} = \int_0^L \int_{-h/2}^{h/2} z^2 \kappa^2 \, dz \, dx = ch^3 \int_0^L \kappa^2 \, dx.$$  

Basic model of paper:

$$\min \int_{g(\Omega)} \kappa_1^2 + \kappa_2^2 \, dA$$

where $\Omega \subset \mathbb{R}^2$, $g : \Omega \to \mathbb{R}^3$ is an isometry, and $\kappa_1, \kappa_2$ are the principal curvatures of the image.

Not necessarily easy to solve (what is the shape of a Mobius band?). Boundary conditions matter; sometimes gravity matters too.
Paper, with singularities

Actually: smooth isometries are not sufficient. Depending on loads and bdry conds, we easily see

- formation of **point singularities** (“d-cones”)

- formation of **line singularities** (“crumpling”)

Why? Because there is no isometry with finite bending energy.

Heuristic calculation: consider a “perfectly conical d-cone”: use
\[ \Omega = \{ r < 1 \} \text{ and } g(r, \theta) = r \varphi(\theta) \text{ where } \varphi(\theta) \text{ is a curve on } S^2 \text{ with the right length } (2\pi). \]
Then

\[
\text{curvature at radius } r \sim \frac{1}{r}
\]

so

\[
\int \kappa^2 \, dA \sim \int_0^1 \frac{1}{r^2} r \, dr \quad \text{is divergent.}
\]
How to model d-cones and crumpling? Main idea:

$$E_h = \min \int_\Omega |(Dg)^T Dg - I|^2 \, dA + h^2 \int_{\text{image}} (\text{curvature})^2 \, dA$$

First term (“membrane energy”) prefers isometry. It is nonconvex (for paper: many smooth minimizers).

Second term (“bending energy”) resists bending, but has $h^2$ in front. If bc permit smooth def with curvature in $L^2$ then $E_h \sim h^2$ as $h \to 0$.

For d-cone and crumpling, $E_h \gg h^2$ as $h \to 0$. (How does $E_h$ behave as $h \to 0$? We have conjectures, but few theorems.)
Digression: folding paper flat

Problem: NYC subway map is difficult to fold “correctly.”

Solution: The Miura map is easy to fold “correctly.”

Is there math here? Well, a little bit:

\[
\text{folding paper flat } \iff g : \mathbb{R}^2 \to \mathbb{R}^2 \text{ such that } Dg \in O(2).
\]

So $g$ is locally a rotation (preserving or reversing orientation).

When 4 creases meet, opposite angles must add to $\pi$ (to permit folding flat).

- For arbitrary “creases” (violating angle condition) there is no way to fold paper flat using them.
- For rectangular creases there are many ways. For Miura pattern there is essentially one way.
In many settings we see wrinkles rather than creases or point singularities. Some examples:

- hanging drapes
- stretched sheets
- water drop on sheet floating on water

As $h \to 0$, scale of wrinkling $\to 0$. (Easiest to see for drapes.)

Mathematically: $E_h = $ stretching energy $+ h^2$ bending energy. But minimizing stretching term requires infinite bending energy.
$E_h = \text{stretching energy} + h^2 \text{bending energy.}$

Min stretching requires infinite bending.

**Problem:** It is difficult to *describe* a pattern (even a well-organized one, as in the figures), let alone explain why it occurs.

**Solution:** Focus instead on the *scaling law* of the minimum energy, i.e. find asymptotics of $E_h$ as $h \to 0$:

$$\text{lower bound} \leq E_h \leq \text{upper bound}$$

- Upper bound can be obtained by guessing form of solution. (Nature often gives us a hint.)
- Lower bound is different: it must consider for *any* pattern (even those not seen in nature).
- If bounds are similar than we have (more or less) nailed it.
A 1D example

\[ \min_{v(0)=v(1)=0} \int_0^1 (v_x^2 - 1)^2 + \varepsilon^2 v_{xx}^2 + \alpha v^2 \, dx \]

- When \( \varepsilon = 0, \alpha > 0 \), min value is 0, not attained. Min sequence has “\( v_x = \pm 1 \) with prob 1/2 each.”
- When \( \varepsilon > 0 \), min scales like \( \varepsilon^{2/3} \alpha^{1/3} \), since for sawtooth with \( N \) teeth, value is about \( \sim \varepsilon N + \alpha N^{-2} \). Best \( N \sim (\alpha/\varepsilon)^{1/3} \).
- Case \( \varepsilon > 0, \alpha = 0 \) is different: just one tooth; min value is \( c_0 \varepsilon \).
2D is richer than 1D

$$\min_{v=0 \text{ at } x=0} \int_{[0,L] \times [0,1]} (v_y^2 - 1)^2 + v_x^2 + \varepsilon^2 |\nabla \nabla v|^2$$

- Like 1D example (in $y$), but microstructure is required by bdry cond rather than a lower-order term.
- Microstructural length scale $\ell$ depends on $x$ (finer near $x=0$). In fact, $\ell(x) \sim \varepsilon^{1/3} x^{2/3}$.
- Min energy has scaling law $\varepsilon^{2/3} L^{1/3}$.
- Studied 20 years ago (Kohn-Müller, Conti); motivated by patterns seen in martensitic phase transformation.

Schematic top view, and photo showing mechanism of refinement:
Recent work with Hoai-Minh Nguyen. Patterns seen in a thin, stiff layer compressed by a thick, soft substrate.

- Stretch a polymer layer
- Deposit the film
- Release the polymer
- Film buckles to avoid compression

Commonly seen pattern: herringbone

silicon on pdms

gold on pdms
We use a “small-slope” (von Karman) version of elasticity, writing \((w_1, w_2, u_3)\) for the elastic displacement. The energy per unit area \(E_h\) has three terms:

1. **Stretching term** captures fact that film’s natural length is larger than that of the substrate:
   \[
   \alpha_m h \int \left| \epsilon(w) + \frac{1}{2} \nabla u_3 \otimes \nabla u_3 - \eta I \right|^2 \, dx \, dy
   \]

2. **Bending term** captures resistance to bending:
   \[
   h^3 \int |\nabla \nabla u_3|^2 \, dx \, dy
   \]

3. **Substrate term** captures fact that substrate acts as a “spring”, tending to keep film flat. Cheating a bit (to simplify), it behaves like
   \[
   \alpha_s \sqrt{\eta} \left( \int u_3^2 \, dx \, dy \right)^{1/2}
   \]
To permit spatial averaging, we assume periodicity on some (large) scale $L$, and we focus on the energy per unit area:

$$E_h = \frac{\alpha_m h}{L^2} \int_{[0,L]^2} |\epsilon(w) + \frac{1}{2} \nabla u_3 \otimes \nabla u_3 - \eta l|^2 \, dx \, dy$$

$$+ \frac{h^3}{L^2} \int_{[0,L]^2} |\nabla \nabla u_3|^2 \, dx \, dy + \frac{\alpha_s \sqrt{\eta}}{L} \left( \int u_3^2 \, dx \, dy \right)^{1/2}$$

Summary of results:

- $\min E_h \sim \min \{ \alpha_m \eta^2 h, \alpha_s^{2/3} \eta h \}$
- energy of unbuckled state $w = u_3 = 0$ is $\alpha_m \eta^2 h$
- energy of herringbone pattern is $C \alpha_s^{2/3} \eta h$.
- herringbone achieves the optimal scaling law in the “far from critical” regime $\alpha_s^{2/3} \ll \alpha_m \eta$. 
Sketch of the upper bound

- Energy of unbuckled state is $\alpha_m \eta^2 h$
- Energy of herringbone is $C \alpha_s^{2/3} \eta h$
- So $\min E_h \leq \min \{\alpha_m \eta^2 h, C \alpha_s^{2/3} \eta h\}$

\[
E_h = \frac{\alpha_m h}{L^2} \int_{[0,L]^2} \left| e(w) + \frac{1}{2} \nabla u_3 \otimes \nabla u_3 - \eta I \right|^2 \, dx \, dy
+ \frac{h^3}{L^2} \int_{[0,L]^2} |\nabla \nabla u_3|^2 \, dx \, dy + \frac{\alpha_s \sqrt{\eta}}{L} \left( \int u_3^2 \, dx \, dy \right)^{1/2}
\]

Key features of herringbone:
- stretching term is negligible
- typical slope is $|\nabla u_3| \sim \sqrt{\eta}$
- if scale of wrinkling is $\ell$ then bending term is $h^3 \eta / \ell^2$ and substrate term is $\alpha_s \eta \ell$. Optimal $\ell \sim h \alpha_s^{-1/3}$ gives value $\sim \alpha_s^{2/3} \eta h$.

Note resemblance to our 1D example

\[
\int_0^1 \left( v_x^2 - 1 \right)^2 + \varepsilon^2 v_{xx}^2 + \alpha v^2 \, dx
\]
Sketch of the lower bound

**Claim:** For any pattern, energy must be at least \(C \min\{\alpha_m \eta^2 h, \alpha_s^{2/3} \eta h\} \).

Let’s take \(L = 1, \eta = 1\) for simplicity. All we need about stretching term is:

\[
\text{stretching term} \geq \alpha_m h \int |\partial_x w_1 + \frac{1}{2}|\partial_x u_3|^2 - 1|^2 \, dx \, dy.
\]

Recall:

\[
\text{bending term} = h^3 \|\nabla \nabla u_3\|_{L^2}^2; \quad \text{substrate term} = \alpha_s \|u_3\|_{L^2}.
\]

Also: \((w, u_3)\) is periodic.

**CASE 1:** If \(\int |\nabla u_3|^2\) is small, stretching \(\geq \alpha_m h\), since \(\partial_x w_1\) has mean 0.

**CASE 2:** If \(\int |\nabla u_3|^2\) is large, use the “interpolation inequality”

\[
\|\nabla u_3\|_{L^2}^2 \leq \|\nabla \nabla u_3\|_{L^2} \|u_3\|_{L^2}
\]

to see that

\[
\text{Bending + substrate terms} = h^3 \|\nabla \nabla u_3\|_{L^2}^2 + \frac{1}{2} \alpha_s \|u_3\| + \frac{1}{2} \alpha_s \|u_3\|^2
\]

\[
\geq C \left(h^3 \|\nabla \nabla u_3\|_{L^2}^2 \alpha_s^2 \|u_3\|^2\right)^{1/3} = Ch\alpha_s^{2/3}.
\]

using arith mean/geom mean inequality.
Claim: For any pattern, energy must be at least $C \min \{\alpha_m \eta^2 h, \alpha_s^{2/3} \eta h\}$.

Let’s take $L = 1$, $\eta = 1$ for simplicity. All we need about stretching term is:

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$$\|\nabla u_3\|_{L^2}^2 \leq \|\nabla^2 u_3\|_{L^2} \| u_3 \|_{L^2}$$

to see that

$$\text{Bending + substrate terms} = h^3 \|\nabla^2 u_3\|^2 + \frac{1}{2} \alpha_s \|u_3\| + \frac{1}{2} \alpha_s \|u_3\| \geq C \left( h^3 \|\nabla^2 u_3\|^2 \alpha_s \|u_3\|^2 \right)^{1/3} = Ch \alpha_s^{2/3}.$$  

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\geq C \left( h^3 \| \nabla \nabla u_3 \|_{L^2}^2 \alpha_s^2 \| u_3 \|^2 \right)^{1/3} = Ch\alpha_s^{2/3}.
\]

using arith mean/geom mean inequality.
Variational viewpoint, with thickness as a small parameter: the bending term is formally smaller than the stretching term by a factor of $h^2$.

Singularities, wrinkling patterns, or both form if minimization of the stretching term requires infinite bending.

Focus on energy scaling law is convenient: at least the question makes sense.

We get insight also about patterns, e.g. whether they achieve the optimal law. (But note: patterns in nature can be local minima.)

Analysis is useful, but so is simulation. Simulation permits exploring local minima, and how patterns form.
Stepping back, cont’d


Other current projects:

1. stretch-induced wrinkling in membranes (Peter Bella)
2. hanging drapes (Peter Bella)
3. confinement-induced wrinkling in floating sheets (Hoai-Minh Nguyen)
4. d-cones (Jeremy Brandman, Hoai-Minh Nguyen)
5. debonding of a compressed film from its substrate (Jacob Bedrossian)


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