These problems are concerned with linear elasticity. Problem 1 asks you to explain why an isotropic Hooke’s law is described by just two constants. Problems 2 and 3 explore Korn’s inequality. Problem 4 gives an example of an elasticity problem with an explicit separation-of-variables solution. Problems 5-8 examine some important reductions and special cases of linear elasticity. There’s also a “food for thought” problem at the very end.

1. **Elastic symmetries.** A linearly elastic material is symmetric under a rotation $R$ if its Hookes’ law satisfies $\alpha(R^T e R) = R^T \alpha(e) R$. Show, by a direct argument, that if this holds for any $R \in SO(3)$ then $\alpha e = 2\mu e + \lambda (\text{tr} e) I$ for some constants $\lambda, \mu$. (Hint: start by showing that $\sigma = \alpha e$ must be simultaneously diagonal with $e$.) What about the case of “cubic symmetry”, when $\alpha$ is only symmetric under 90 deg rotations (i.e. under any $R$ which permutes the coordinate axes)?

2. **Korn’s inequality for periodic deformations.** Korn’s inequality for periodic deformations says

$$ \int_Q |\nabla u|^2 \, dx \leq C \int_Q |e(u)|^2 \, dx $$

when $u : \mathbb{R}^n \to \mathbb{R}^n$ is periodic in each variable with period 1 and $Q = [0,1]^n$ is the unit cell. Give a proof using the Fourier representation of $u$. What is the best possible value of the constant $C$? Why is there no condition about $\int \nabla u$ being symmetric?

3. **Korn’s inequality for beams.** Let $\Omega_h \subset \mathbb{R}^2$ be the long, thin domain $\{0 < x < 1, -h/2 < y < h/2\}$ where $h \ll 1$. Korn’s second inequality for this domain says

$$ \int_{\Omega_h} |\nabla u|^2 \, dx \leq C(h) \int_{\Omega_h} |e(u)|^2 \, dx \quad \text{provided } \int_{\Omega_h} \nabla u \text{ is symmetric.} $$

(a) Show that $C(h)$ must be at least of order $h^{-2}$, by considering deformations of the form $u(x,y) = (-y\phi_x, \phi)$ where $\phi = \phi(x)$.

(b) Show that the inequality is true with $C_h \sim h^{-2}$. You may assume (for simplicity, this is not really necessary) that $1/h$ is an integer. Hint: divide $\Omega_h$ into $1/h$ squares of side $h$. Korn’s inequality (for squares) controls $\nabla u - \begin{pmatrix} 0 & \omega_j \\ -\omega_j & 0 \end{pmatrix}$ on the $j$th square in terms of the strain on that square, for some $\omega_j \in R$. Use Korn’s inequality again (this time for rectangles of eccentricity 2) to control $\omega_j - \omega_{j-1}$ in terms of the strain on the $(j-1)$st and $j$th squares. Then apply a discrete version of Poincare’s inequality in one space dimension to control the variation of $\omega_j$ with $j$.

(c) How do you think these results would extend to a thin plate-like domain $\{0 < x < 1, 0 < y < 1, -h/2 < z < h/2\}$ in $\mathbb{R}^3$? (Just discuss how the 3D problem is similar or different; I’m not asking for a complete solution.)

[Comment: The proof of the nonlinear Korn inequality in the paper by Friesecke, James, and Müller uses the strategy outlined in (b) above, applied to the nonlinear rather than the linear setting.]
4. Separation of variables. Let Ω be a “ball with a hole removed”:

\[
\Omega = \{ x : \rho^2 < |x|^2 < 1 \}.
\]

Suppose it is filled with an isotropic, homogeneous, linearly elastic material with Lamé moduli \( \lambda \) and \( \mu \), and constant pressure \( P \) is applied at the outer boundary \( |x| = 1 \). The inner boundary \( |x| = \rho \) is traction-free. Find the displacement \( u(x) \) and the associated stress \( \sigma(x) \) using separation of variables.

5. The torsion problem. Let \( D \) be a domain in the \( x - y \) plane, and consider a long cylinder with cross-section \( D \). Imagine twisting the cylinder at its ends. The lateral boundaries are traction-free, and gravity is ignored. The linearized version of such a deformation is achieved by

\[
u(x, y, z) = \tau(-yz, xz, \phi(x, y))
\]

for \( \tau \in \mathbb{R} \) and \( \phi : D \rightarrow \mathbb{R} \).

(a) Find the associated stress and strain, assuming an isotropic and homogeneous Hooke’s law. Show that \( u \) solves the equations of elastostatics with traction-free boundary condition \( \sigma \cdot n = 0 \) at the lateral boundaries (and a suitable displacement boundary condition at the ends) if and only if \( \Delta \phi = 0 \) in \( D \) and \( \partial \phi / \partial n = (y, -x) \cdot n \) at \( \partial D \).

(b) Verify that the consistency condition \( \int_{\partial D} (y, -x) \cdot n = 0 \) is satisfied [thus \( \phi \) exists and is unique up to an additive constant].

(c) Show that the elastic energy per unit length is \( \tau^2 T \) where \( T = \mu \int_D (\phi_x - y)^2 + (\phi_y + x)^2 \, dx \, dy \). This \( T \) is called the torsional rigidity of the cylinder.

[Comment: This example is more than just a special solution: “Saint Venant’s principle” says that no matter how you twist the ends of a cylinder, far from the ends the deformation will approach the special solution described above.]

6. Antiplane shear. Consider once again a cylinder with cross-section \( D \), but consider a uniform body load in the \( z \) direction (gravity), and suppose the lateral boundaries are clamped. Show that these conditions are consistent with the displacement \( u = (0, 0, \phi(x, y)) \) with \( \Delta \phi = 1 \) in \( D \) and \( \phi = 0 \) at \( \partial D \).

7. Bending of a thin plate. Consider now a thin, constant-thickness plate whose midplane occupies a region \( D \) in the \( x - y \) plane. The upper and lower surfaces are \( z = \pm h/2 \), so the thickness is \( h \). Consider a deformation of the form \( u = (z\phi_y - z\phi_x, \phi + \frac{\alpha}{2} z^2 \Delta \phi) \). Find the associated strain and stress, keeping only terms of order \( h \). Show that for the faces to be traction-free (to this order) we need \( \alpha = \lambda / (\lambda + 2\mu) \). Do the \( z \)-integrations in the basic variational principle, to obtain a new variational principle for \( \phi(x, y) \). Notice that it involves second derivatives of \( \phi \), so the associated PDE is a fourth-order equation!
8. **Plane stress.** Consider the same thin plate, but rather than bending it we suppose it is loaded within its plane. The top and bottom are traction-free, so \( \sigma_{33} = 0 \) there. If the plate is thin enough we may expect that \( \sigma \) is independent of \( z \). This does not imply that \( u_i \) are independent of \( z \), but we can nevertheless consider \( \bar{u}_i(x,y) \) = the average of \( u_i \) with respect to \( z \). Show that \( \bar{u}_1, \bar{u}_2 \) solve the system of “2D elasticity” with a suitable choice of elastic constants.

This problem set is long enough. Therefore I’m not asking you to write up the following “food for thought” problem, which we mentioned briefly in class. It addresses a fundamental point about the link between PDE and variational principles. Be sure you understand it.

**Food for thought problem.** One might think you could solve

\[
\Delta u + f = 0 \text{ in } \Omega, \quad \partial u / \partial n = g \text{ at } \partial \Omega
\]

by minimizing \( \int_\Omega \frac{1}{2} |\nabla u|^2 - uf \) subject to the constraint \( \partial u / \partial n = g \) at \( \partial \Omega \). Show this is false unless \( g = 0 \), by considering the case when \( \Omega = [0,1] \) is an interval, and demonstrating the following assertions:

(a) The constrained variational problem is unbounded below if \( \int_0^1 f \neq 0 \).

(b) If \( \int_0^1 f = 0 \) then the constrained variational problem is bounded below, and its minimum value is the same is if the constraint were not present.

(c) Still assuming \( \int_0^1 f = 0 \), a minimizing sequence for the constrained variational problem converges to a (weak) solution of \( \Delta u + f = 0 \) in \((0,1)\) with \( u_x = 0 \) at the endpoints \( x = 0 \) and \( x = 1 \).