These problems provide practice with basic concepts of 3D nonlinear elasticity, and explore various reductions including (i) incompressible fluid dynamics, (ii) elastic membranes, and (iii) balloons.

(1) A homogeneous elastic fluid is a hyperelastic material with an energy function $W(F) = h(\det F)$. Show that the Cauchy stress is then $\tau = -p(\rho)I$, where $p(\rho) = -h'(\rho R)/\rho$. [Here $\rho R$ is the density in Lagrangian, assumed constant, and $\rho$ is the density in Eulerian variables.] Show that in this case the equations of elastodynamics are precisely the compressible Euler equations

$$\rho \left( \frac{\partial v}{\partial t} + v \cdot \nabla v \right) = -\nabla p(\rho) + \rho f$$

$$\frac{\partial p}{\partial t} + \sum \frac{\partial}{\partial x_i}(\rho v_i) = 0.$$ 

[Note: to calculate $\frac{\partial W}{\partial F_i^{\alpha}}$ when $W(F) = h(\det F)$ you'll to use Cramer’s Rule, which says that $\frac{\partial}{\partial F}(\det F) = (\det F)(F^T)^{-1}$.

(2) Consider a hyperelastic material, whose Piola-Kirchhoff stress tensor is given by $P_{i\alpha} = \frac{\partial W}{\partial F_i^{\alpha}}$. Show that if $W$ is frame-indifferent (i.e. if $W(F) = W(RF)$ for all orientation-preserving rotations $R$) then the associated Cauchy stress $\tau$ satisfies $\tau(RF) = R\tau(F)R^T$.

(3) Consider a homogeneous, isotropic, hyperelastic material with energy function $W(F) = \psi(I_1, I_2, I_3)$, where $I_1, I_2, I_3$ are the elementary symmetric functions of $B = FF^T$ ($I_1 = \text{tr} B$, $I_2 = \frac{1}{2}[(\text{tr} B)^2 - \text{tr}(B^2)]$, $I_3 = \det B$). Show that the associated Cauchy stress has the form $\tau = \phi_0 I + \phi_1 B + \phi_2 B^2$ with

$$\phi_0 = 2\frac{\partial \psi}{\partial I_3} \det F$$

$$\phi_1 = 2\frac{\partial \psi}{\partial I_1}(\det F)^{-1} + 2\frac{\partial \psi}{\partial I_2}(\text{tr} B)(\det F)^{-1}$$

$$\phi_3 = -2\frac{\partial \psi}{\partial I_2}(\det F)^{-1}.$$ 

(4) Rubber can be modelled as a homogeneous, isotropic, incompressible hyperelastic material. The energy function for such a material has the form $W(F) = \psi(I_1, I_2)$, since all deformations must satisfy the constraint $\det F = 1$. Its Cauchy stress has the form $\tau = -pI + \phi_1 B + \phi_2 B^2$, where $\phi_1, \phi_2$ have the form derived in Problem 3. Let’s explore how $W$ can be determined experimentally, using relatively simple experiments on thin membranes.

Consider a sheet (in reference coordinates) of length $2A$, width $2B$, and thickness $2h$, with $A, B \gg h$. Consider deformations of the form

$$x_i = \lambda_i X_i, \quad i = 1, 2, 3,$$

which can be maintained by edge tractions alone (i.e. for which the the faces $X_3 = \pm h$ are traction-free). Show that

$$I_1 = \frac{\lambda_1^2 + \lambda_2^2}{\lambda_3} + \frac{1}{\lambda_1^2 \lambda_2^2}$$

$$I_2 = \frac{1}{\lambda_1} + \frac{1}{\lambda_2} + \lambda_1^2 \lambda_2^2$$
and that the Cauchy stress is 
\[
\begin{align*}
\tau_{11} & = 2(\lambda_1^2 - \frac{1}{\lambda_1^2 \lambda_2^2})(\frac{\partial \psi}{\partial I_1} + \lambda_2^2 \frac{\partial \psi}{\partial I_2}) \\
\tau_{22} & = 2(\lambda_2^2 - \frac{1}{\lambda_1^2 \lambda_2^2})(\frac{\psi}{\partial I_1} + \lambda_1^2 \frac{\psi}{\partial I_2}) \\
\tau_{33} & = 0 \\
\tau_{ij} & = 0 \quad i \neq j.
\end{align*}
\]

Conclude that \(\frac{\partial \psi}{\partial I_1}\) and \(\frac{\partial \psi}{\partial I_2}\) satisfy
\[
\begin{align*}
\frac{\partial \psi}{\partial I_1} & = \frac{1}{2(\lambda_1^2 - \lambda_2^2)} \left( \frac{\lambda_1^2 \tau_{11}}{\lambda_1^2 - 1/\lambda_1^2 \lambda_2^2} - \lambda_2^2 \frac{\lambda_2^2 \tau_{22}}{\lambda_2^2 - 1/\lambda_1^2 \lambda_2^2} \right) \\
\frac{\partial \psi}{\partial I_2} & = \frac{-1}{2(\lambda_1^2 - \lambda_2^2)} \left( \frac{\tau_{11}}{\lambda_1^2 - 1/\lambda_1^2 \lambda_2^2} - \lambda_2^2 \frac{\tau_{22}}{\lambda_2^2 - 1/\lambda_1^2 \lambda_2^2} \right).
\end{align*}
\]

Thus by measuring the dependence of \(\tau_{11}\) and \(\tau_{22}\) on \(\lambda_1\) and \(\lambda_2\) one can determine the function \(\psi\).

(5) Consider a spherical rubber balloon (such as you might buy in a toy store). To a reasonable approximation we may:

- consider the reference domain to be a thin spherical annulus \(\Omega = \{x : r_0 - \epsilon < |X| < r_0 + \epsilon\}\);
- consider the air pressure in the balloon to be a constant \(p\);
- ignore the atmospheric pressure outside the balloon;
- consider experiments that are volume-controlled (fixing the volume of the interior of the balloon) or pressure-controlled (fixing the air pressure in the balloon).

From common experience, it is difficult to start blowing up a balloon, but then it gets easier, though eventually as the balloon gets large the blowing gets hard again (unless it bursts). This suggests a pressure-volume relation of the type shown in figure 1 below.

(a) Assume the rubber is hyperelastic and incompressible. Show that variational principle associated with a pressure-controlled experiment involves the energy \(E = \int_\Omega W(F) \, dX - p(\text{volume inside balloon})\). (In other words, check that this gives the correct equilibrium and boundary conditions.) What variational principle is associated with a volume-controlled experiment?

(b) Consider the limit \(\epsilon \to 0\) and assume the deformation is uniform expansion (i.e. the sphere \(X = r_0\) is mapped by \(x(X) = \lambda X\) to a sphere of radius \(\lambda r_0\)). Suppose \(W\) has the form \(\Phi(\lambda_1, \lambda_2, \lambda_3)\) where \(\lambda_1, \lambda_2, \) and \(\lambda_3\) are the principal stretches (eigenvalues of \((F^T F)^{1/2}\)). Show that when restricted to the case of “uniform expansion” the pressure-controlled variational principle takes the form \(E(\lambda) = c_1 F(\lambda) - c_2 p \lambda^3\) with \(F(\lambda) = \Phi(\lambda, \lambda, \lambda^{-2})\).

What are the constants \(c_1\) and \(c_2\)?
(c) Two commonly-used constitutive laws for rubber are the \textit{neo-Hookean} energy

$$\Phi(\lambda_1, \lambda_2, \lambda_3) = a(\lambda_1^2 + \lambda_2^2 + \lambda_3^2 - 3)$$

with \(a > 0\), and the \textit{Mooney-Rivlin} energy

$$\Phi(\lambda_1, \lambda_2, \lambda_3) = a(\lambda_1^2 + \lambda_2^2 + \lambda_3^2 - 3) + (a/K)(\lambda_1^{-2} + \lambda_2^{-2} + \lambda_3^{-2} - 3)$$

with \(a > 0\) and \(K > 0\) (typically \(4 < K < 8\)). Are these laws consistent with the nonmonotone pressure-volume relation shown in figure 1?

(d) Let’s think about the 1D energy \(E(\lambda)\), using the non-monotonicity of the pressure-volume relation (as shown in Figure 1) but not using any special formula for \(F\) (such as those in part b). Evidently, certain values of the pressure \(p\) are consistent with 3 different volumes rather than just one. For such \(p\), \(E\) must have “double-well” structure, as shown in Figure 2. Show that the two wells have exactly the same depth precisely when \(p = p_0\) satisfies the “equal area rule” sketched in Figure 3.

(e) In real pressure-controlled experiments, as \(p\) crosses the value \(p_0\), the balloon size changes (relatively suddenly) so that the volume occupies the deeper well (the energetically preferred state). How can this be reconciled with our 1D model?