Mechanics - Lecture 6, 2/29/2012

(Start 2/29 by finishing the “Lecture 6” notes, introducing linear elasticity.)

Remaining task: discuss existence + uniqueness for Laplacian of linear elastostatics:

eqns: $\text{div}\sigma + f = 0 \rightarrow \sigma = 2\varepsilon(u)$ in $\Omega$

bc $u = u_0$ (displ bc) or $\sigma \cdot n = f$ (traction bc)

Discussion parallels in many ways the existence + uniqueness of solutions to Laplace's eqn (or more generally, divergence-free scalar eqn $\nabla \cdot (\mathbf{a}(x)\nabla u) = f$ with Dirichlet or Neumann bc's in scalar-valued $\mathbf{a}(x)$).

Major difference is that we need Kantorovich's inequality in place of Poincaré's inequality.

First let's discuss uniqueness:

- In nonlinear elasticity, uniqueness is false even when displacement is specified on entire body. Classical thought-expt due to E. John: 2D annulus. Imagine twisting inner circle by $2\pi$. Should get soln $x(X) = X$ on entire body, but $x(X) \neq X$ inside (see figure).
By contrast, we'll show that only of linear elasticity, with displacement $bc$ is unique.

- In nonlinear elasticity, buckling demonstrates nonuniqueness for planar axi-symmetric beam or column, where beam is clamped type or part of $\partial \Omega$ + traction-free or rest

\[ \begin{array}{c}
\text{Clamp} \\
bc
\end{array} \xrightarrow{\text{buckling}} \begin{array}{c}
\text{Traction-free}
\end{array} \]

By contrast, we'll show that only of linear elasticity is unique if displ $bc$ is specified on any part of $\partial \Omega$.

- Recall simple proof of uniqueness for $\Delta u = f$ in $\Omega$, $u = u_0$ at $\partial \Omega$; if $u$ two reals, their difference solves $\Delta u = 0$ in $\Omega$, $u = 0$ at $\partial \Omega$. Multiply by $u$ and integrate by parts: $\int u \Delta u = \frac{1}{2} \int u^2$. $\Rightarrow \int u = 0 \Rightarrow u = 0$ (by bc).
Similar arg works for elastostatics: consider \( \Omega \)

\[ \delta u(x, e(u)) = \int_{\Omega} \nabla^2 \nabla \cdot \nabla e(u) \, dx \]

where \( \partial \Omega = \Gamma_1 \cup \Gamma_2 \). So \( u \) is unique.

If \( f = g = 0 \), \( u_0 = 0 \) \( \Rightarrow u \equiv 0 \). Argue as for Laplace: with \( \sigma = x e(u) \),

\[ 0 = \int_{\Gamma_1} \langle u, \delta u \rangle \, d\Gamma = -\int_{\Gamma_2} \langle u, \delta u \rangle \, d\Gamma \]

by Green's

\[ = -\int_{\partial \Omega} \langle u, \sigma \rangle \, d\Gamma \]

since \( \sigma \) is symmetric

\[ = -\int \langle x e(u), e(u) \rangle \, d\sigma \]

\( \Rightarrow e(u) \equiv 0 \) since \( x \) is positive.

Now need **Lemma**: if \( e(u) \equiv 0 \) in connected region \( \Omega \)
then \( u \) is an "invol rigid motion" i.e.

\[ u(x) = \sum \omega_j x_j + d_i \]

for some (constant) skew-symmetric \( \omega_{ij} \) and some constant \( d_i \).
(Note: This is linear analogue of stiff that $F^{-1}F = I \Rightarrow x(X) is locally a rigid motion.)

Proof is easy in $\mathbb{R}^2$:

\[ u_{11} \equiv 0, \quad u_{22} \equiv 0 \Rightarrow u_1 = f(x_2), \quad u_2 = f(x_1) \]

\[ u_{12} + u_{21} = 0 \Rightarrow f'(x_2) + f'(x_1) = 0 \]

\[ \Rightarrow f = \omega x_2 + \text{const} \]

\[ q = -\omega x_1 + \text{const} \]

Proof in $\mathbb{R}^3$ ($\mathbb{R}^2, n \geq 3$) can be done similarly, by induction on dimension. Or, there's another (less intuitive) argument: observe that

\[ \partial_k u_i = \partial_k e_i + \partial_k e_j - \partial_i e_j \]

Therefore $e_i(x) \equiv 0 \Rightarrow \nabla \nabla u = 0 \Rightarrow u$ is linear in $x$. Since $e_i(x) \equiv 0$, $D u$ is skew-symmetric.

Wrap up proof of uniqueness: we were assuming $d\sigma = 0$ in $\Sigma$, $u = 0$ at $\Gamma'$, $\sigma_n = 0$ at $\Gamma_2$. We concluded $u = \text{unif rigid motion}$.

If $\Gamma' \neq \emptyset$ this forces $u = 0$.

What about pure traction problem? Situation
is like Neumann plan to Laplace eqn.
Recall: for \( \Delta u = f \) in \( \Omega \), \( \frac{\partial u}{\partial n} = g \) at \( \partial \Omega \) we have a **consistency condition** \( \int \int_\Omega g \frac{\partial u}{\partial n} \, d\Omega = \int \int_\Omega f \, d\Omega \) when consistency holds, \( u \) is unique up to a constant.

**Similar situation in linear elasticity**:
- \( \mu \Delta \sigma = f \), \( \sigma = 2 \varepsilon(u) \) in \( \Omega \)
- \( \sigma \cdot n = g \) at \( \partial \Omega \)

can have only if

\[
\int_\partial \Omega \langle g, \hat{u} \rangle \, ds + \int_\Omega \langle f, \hat{u} \rangle \, dx = 0
\]
wherever \( \langle \cdot, \cdot \rangle \) is an inner product, \( \hat{u} \) is an unit rigid motion.

If it does have a \( u \), that \( u \) is unique up to addition of an unit rigid motion.

**Proof of consistency**: \( c(u) = 0 \) then
\[
\int_\partial \Omega \langle g, \hat{u} \rangle \, ds = \int_\Omega \langle \hat{u}, \Delta \sigma \rangle \, dx = -\int_\Omega \langle c(\hat{u}), \varepsilon \rangle \, dx
\]
\[
+ \int_\partial \Omega \langle \hat{u}, \sigma \cdot n \rangle \, ds
\]
If of uniqueness: same as before except now \( \Omega = \emptyset \) so \( u \) can be a nonzero constant.

What about existence? Again, again is a lot like scalar Laplace eqn, or div-form scalar eqn \( \nabla \cdot (\sigma(x) \nabla u) = f \). Main techniques:

1. Variational principles \( \Rightarrow \) very closely connected!
2. Lax-Milgrorn lemma
3. Babuška integral techniques, (different)

Babuška integral methods are basically restricted to constant-coefficient setting (won’t discuss them here).

Variational and Lax-Milgrorn are simple & general; also form basis of most numerical schemes (e.g. finite elements), we’ll focus on former.

Again, use scalar Laplace as guide. Solve to

\[ \Delta u = f \text{ in } \Omega, \quad u = u_0 \text{ on part of } \partial \Omega \ (1') \]
\[ \frac{\partial u}{\partial n} = g \text{ on rest of } \partial \Omega \ (1'') \]

can be found using variational prin
\[
\min \int \frac{1}{2} \nabla u^2 + f u \, dx - \frac{1}{2} \int u^2 \, dx
\]

\text{on } \Gamma_0 \text{ and } \Gamma, \forall x \in \Omega.

\text{if } \Gamma = \emptyset \text{ and data are inconsistent, then functional is unbounded below; we can drive it to } -\infty \text{ by taking } u = \text{suitable constant}.

Existence via variational means convexity of \( \mathcal{L}_0 \), plus lemma that it's odd below. Key to latter is a part of Poincaré-type inequalities.

\text{Poincaré inequality: } \int_{\Omega} |u|^2 \leq C \int_{\Omega} |\nabla u|^2

\text{provided } u = 0 \text{ at } \partial \Omega.

\text{and Poincaré inequality: } \int_{\Omega} |u - \bar{u}|^2 \leq C \int_{\Omega} |\nabla u|^2

\text{where } \bar{u} = \text{avg of } u \text{ on } \Omega.

They assure us that although "energy" controls only \( \int |\nabla u|^2 \) directly, it also controls \( \int |u|^2 \) indirectly.

Situation for elasticity is just the same. Vaillami proved
\[
\begin{align*}
\min_{u \in H^1_0(\Omega)} & \quad \int_{\Omega} \frac{1}{2} \langle \Delta u, u \rangle \, dx + \langle f, u \rangle \, dx - \frac{1}{2} \langle g, u \rangle \\
= & \quad \frac{1}{2} \int_{\Omega} \frac{1}{2} \langle \Delta u, u \rangle \, dx + \langle f, u \rangle \, dx - \frac{1}{2} \langle g, u \rangle
\end{align*}
\]

Analogue of Poincaré inequality is **Korn's inequality**: 

**Easy Korn inequality**:

\[
\int_{\Omega} \frac{1}{2} \langle \nabla u, u \rangle \, dx \leq C \int_{\Omega} |\varepsilon(u)|^2 \, dx
\]

provided \( u = 0 \) at \( \partial \Omega \)

**Hard Korn inequality**:

\[
\int_{\Omega} \frac{1}{2} \langle \nabla u, u \rangle \, dx \leq C \int_{\Omega} |\varepsilon(u)|^2 \, dx
\]

provided \( \int_{\Omega} |\varepsilon(u)| \) is symmetric matrix.

These ensure that although "energy" controls only \( \int |\varepsilon(u)|^2 \) directly, it controls \( \int |\nabla u|^2 \) indirectly (and therefore also \( \int |u|^2 \) indirectly).

Some intuition on (hard) Korn inequality: it clearly implies

\[
\int_{\Omega} |u - \hat{u}|^2 \leq C \int_{\Omega} |\varepsilon(u)|^2 \quad \text{for some underlaid rigid rotation}
\]

which is linear analogue of estimate (true, but much harder) that [small random strain] \( \Rightarrow [\text{close to rigid rotation}] \). Const depends on domain, of course,
and long, thin domains \implies very large constants.

\[ \rightarrow \]

locally close to a rigid rotation.

But not globally!

"Easy Korn way" can be proved by an elementary integral by parts, or by an easy Fourier-transform-based argument. Here is the former:

For \( u \in C^0_c(\mathbb{R}) \) (with \( \mathbb{R} = \mathbb{R}^d \)),

\[ \frac{1}{2} \sum \left( \frac{\nabla u_i + \nabla u_i^T}{2} \right)^2 \]

\[ = \frac{1}{2} \sum \frac{1}{2} |\nabla u_i|^2 + \frac{1}{2} \sum \nabla u_i \cdot \nabla u_i \]

But since \( u = 0 \) near \( \partial \Omega \),

\[ \frac{1}{2} \sum \nabla u_i \cdot \nabla u_i = \frac{1}{2} \sum \nabla u_i \cdot \nabla u_i \]

So

\[ \frac{1}{2} \sum |\nabla u_i|^2 = \frac{1}{2} \int_{\partial \Omega} \left| \nabla u_i \right|^2 + \frac{1}{2} \int_{\Omega} \left| \nabla u_i \right|^2 \]
\[ \leq C J \sqrt{u^2} \]

Result extends to all \( u \in H_0^1(\Omega) = \{ u \in H^1(\Omega), u = 0 \text{ on } \partial \Omega \} \) by compactness.

"Hard Korn map" has interesting history:

1st proof by Korn, about 1910.

Friedrichs wrote paper abt 1947 pointing out importance of this map, giving a new (still not really simple) proof, and "modern" treatment of existence theory.

Many proofs since Friedrichs. Some very efficient but not so elementary (see eg. Duvaut + Lions book).

Too much study of other "coerciveness map"; how does control of selected few controls of \( u \) yield control of all domains individually? (KT Smith, Amazigh, others - 60's+70's, using "pseudodifferential operators")

A really simple, elementary proof was finally given by Oleinik + Kondratiev about 1989 (CRAS Paris Sec 1, 1989, 483-487; also Rev. Mat. Appl. 10, 1999, no 3, 641-666). Separate handout gives (essentially) their proof.