We've been doing "fully nonlinear" elasticity.
Now let's turn to linear elasticity. This entails two separate linearizations:

1. Geometrical linearization ("small strain theory") - we suppose

\[ X_{\delta}(X) = X + \delta u(X) \]

with \( \delta \ll 1 \). Here \( u \) is the "infinitesimal elastic displacement." Evidently

\[ F = I + \delta : Du \]

\[ \Rightarrow (F^TF)^{1/2} = I + \frac{\delta}{2}(Du + Du^T) + O(\delta^2) \]

so to leading order, \((F^TF)^{1/2} - I\) is \( \delta \times \) times the linear elastic strain

\[ e(u) = \frac{1}{2}(Du + Du^T) \]
2nd physical linearization ("linear stress-strain law")

Using hyperelasticity as a starting point (spatially homogeneous, for simplicity) with \( W(0) = 0 \) and \( W \geq 0 \) (to net stable is a local min of elastic energy), we have

\[
W(F) = \delta^2 \langle \varepsilon(F), \varepsilon(F) \rangle + \mathcal{O}(\delta^3)
\]

where

\[
\langle \varepsilon, \varepsilon \rangle = \sum_{ijkl} \varepsilon_{ijkl} \varepsilon_{ijkl}
\]

is a pos def (symmetric) quadratic form on symmetric tensors.

Dropping terms of higher order in \( \delta \) (and rescaling by \( 1/\delta^2 \)), we expect a weak form of the form \( \delta E = 0 \) where

\[
E = \frac{1}{2} \int \langle \varepsilon(e(u)), \varepsilon(e(u)) \rangle \, dx + \text{[terms assoc to loads]}
\]

so equal \( \varepsilon / \rho \) is

\[
\text{div}(\delta \varepsilon(e(u))) + f = 0
\]
and we recognize
\[ \sigma = \lambda e(u) \]
as the stress tensor in the linear elastic setting.

Notes:
1) $\sigma$ is symmetric.
2) We lose the ability to distinguish between rest and deformed configurations when we linearize.

(Put differently: the Cauchy + Piola-Kirchhoff stresses agree to 1st order in $\varepsilon$; that's why $\sigma$ is symmetric.)

Summary: in linear elasticity, basic unknown is "displacement" $u(x)$. Associated strain is $e(u) = \frac{1}{2} (Du + Du^T)$. Stress is $\sigma = \lambda e(u)$. Equilibrium are
\[ \text{div } \sigma + f = 0, \quad \sigma = \lambda e(u) \]
augmented by suitable BC, for example
\( \nabla \cdot u = 0 \) at $\partial \Omega$ ("displacement BC")
\( \partial \nabla \cdot \sigma = g \) at $\partial \Omega$ ("traction BC")
\[ \varepsilon \cdot \mathbf{n} = 0, \quad (\sigma \cdot \mathbf{n}) = 0 \quad \text{at } \mathcal{D} \]

Lubricated 6c

We discussed isotropy in nonlinear setting. Anisotropy for linear elasticity is


Hooke's law \( \varepsilon \Rightarrow \sigma(R^T e R) = R^T (\varepsilon \cdot e) R \)

is isotropic for any \( \varepsilon \) (symmetric) and any \( R \) (rotation).

(More generally: \( R \) is a symmetry of the material if \( \sigma(R^T e R) = R^T (\varepsilon \cdot e) R \) for any \( \varepsilon \). Intuition: rotating material \( \Rightarrow \) keep material fixed but rotate coordinates.)

Lemma: The general isotropic Hooke's law can be expressed as

\[ \sigma_{ij} = 2\mu \varepsilon_{ij} + \lambda (\varepsilon : \varepsilon) \delta_{ij} \]

where \( \lambda + 2\mu \) are constants ("Lamé moduli").

Example (not quite a proof): Hooke's law isotropic energy \( \langle \varepsilon \cdot e, e \rangle \) is a quadratic, isotropic...
function of symmetric tensor $e_{ij}$. Fact of linear algebra: all such $e_{ij}$ are obtained by starting from $e_{ij} e_{ik}$

and "contracting indices" in pairs. There are two distinct ways to do this:

a) $\sum_{i,j} e_{ij} e_{ik} = (\text{tr} e)^2$

b) $\sum_{i,j} e_{ij} e_{ik} = \lambda (e^2) = 1 e_1^2$

So,

$\langle e, e \rangle = 2 \lambda e_1^2 + \lambda (\text{tr} e)^2$

For some $\lambda + \mu$.

There are other common repres of an isotropic Hooke's law, and good reasons to use them:

1st in terms of Young's modulus $E$ and Poisson's ratio $\nu$:

$$\sigma_{ij} = \frac{E}{1 + \nu} \left( e_{ij} + \frac{\nu}{1 - 2\nu} (\text{tr} e)^{\delta_{ij}} \right)$$
in terms of bulk modulus $K$ and shear modulus $\mu$,

$$\sigma_{ij} = 3K \left( \frac{1}{3} \delta_{ij} \right) + 2\mu \left( e_{ij} - \frac{1}{3} \left( \text{tr} e \right) \delta_{ij} \right)$$

Positively requires

\[\begin{align*}
\mu &> 0 \\
3\lambda + 2\mu &> 0
\end{align*}\]

For plane moduli for Young's modulus, Poisson's ratio, for bulk + shear moduli.

Of course one set of parameters determines the others, e.g.

$$\lambda = \frac{E\nu}{(1-\nu)(1-2\nu)} \quad \mu = \frac{E}{2(1+\nu)}$$

$$\lambda = K - \frac{1}{3} \mu$$

Meaning of these parameters:

- Bulk + shear moduli describe action of $\sigma_{ij}$ on multiples of identity + trace-free matrices ("pure shears") separately.

$$\sigma_{ij} = 3K \left( \frac{1}{3} \delta_{ij} \right) + 2\mu \left( e_{ij} - \frac{1}{3} \left( \text{tr} e \right) \delta_{ij} \right)$$

\[\text{proj of } e \text{ onto mult of } I \quad \text{proj of } e \text{ onto } I\]
Note: bulk modulus measures vol change due to hydrostatic pressure.

Shear modulus gives response to any pure shear, e.g. \( \sigma_{12} = T \), \( \sigma_{ij} = 0 \) otherwise.

\[ e_{12} = \frac{1}{2\mu} \sigma_{12} \]

Poisson's ratio + Young's modulus describe behavior under uniaxial tension (\( \sigma_{11} = T \), other \( \sigma_{ij} = 0 \)).

\[ e_{11} = \frac{1}{E} T \]

\[ e_{22} = e_{33} = -\frac{v}{E} T \]

Most materials have \( v > 0 \), so uniaxial tension produces contraction in the earthy vars. Cork has \( v = 0 \), which is why it is used for closing wine bottles.

There are lots of scalar or 2D reductions of
elastostatics that help generate intuition: see e.g. Howells, Kozyreff, Ockenden on

anti-plane strain (Sec 2.3) [ scalar reductions
extension (Sec 2.4, 2.5) ]

plane strain (Sec 2.6) [ a second-order system, but
equals to a 4th order scalar pde

We'll turn in next lecture to a brief discussion of
existence + uniqueness (via Korn’s inequalities +
variational principles).}