Continuing 3D nonlinear elasticity. Discussed so far: strain + stress. Remaining tasks:

a) constitutive laws (relate strain + stress)
b) "reference" vs "deformed" coordinates (i.e. Lagrangian vs Eulerian)
c) examples

We'll do (a) + (b) in this lecture (they're strongly coupled); the next HW will cover (c).

Why are (a) + (b) so connected? Because

A) variational principles are easiest to state + solve in ref. (Lagrangian) coordinates, but

B) measurements + most typical loads (e.g. pressure) usually come to us as force per unit deformed area

Explaining (A): We'll see that a convenient approach to constitutive modeling is to insist that:

elastostatics \( \Rightarrow \) \( \mathbf{S} \mathbf{E} = 0 \), where

\[
E = \int_{\mathcal{L}} W(\mathbf{D} \mathbf{x}) \, d\mathbf{x} + \int_{\mathcal{L}} V(\mathbf{S} \mathbf{x}) \, d\mathbf{x}
\]
with $\mathbf{E}$ = reference config + $W(F)$ a $n \times n$ matrix ("elastic energy density") satisfying certain structural conditions. Here pole

$$\sum_{x=1}^{3} \frac{1}{E_{x}} \left[ \frac{\partial W}{\partial E_{x}} \right] \left( \frac{\partial x}{\partial x} \right) - \frac{\partial V}{\partial x} (x(X)) = 0$$

expresses balance of forces in ref. corrdl.

$$\frac{\partial W}{\partial E_{x}} (x(X)) = \text{force in dirn i per unit ref area, acting on surf.}$$

$$= \text{1st Peola-Kirchhoff stress tensor} \, \mathbf{P}_{i\alpha}$$

$$- \frac{\partial V}{\partial x} (x(X)) = \text{force per unit ref vol}$$

acting at location $x(X)$.

Ventric is convenient because we're solving a pole on a fixed domain $\mathcal{X}$.

**Explaining (B)**: Recall that

$$\mathbf{T}_{ij} = \text{force in dirn i per unit def. area, acting on surf.}$$

$$\text{1st coord vector in deformed coords}$$
and that \( \tau \) is symmetric (\( P_{in} \) is not symmetric.)

A common be case is "constant pressure \( P_0 \)" which means

\[
\tau \cdot \bar{n} = -P_0 \bar{n} + \partial x(\Omega),
\]

where \( \bar{n} \) = unit normal to \( \partial x(\Omega) \). Simple to say in Euclidean cooncepts, messy to write in Lagrang coordinates.

"Traction free" be case is easy in both concepts

\[
\tau \cdot \bar{n} = 0 \text{ at } \partial x(\Omega), \quad \bar{n} = \text{unit normal to } \partial x(\Omega)
\]

\[
P \cdot N = 0 \text{ at } \partial \Omega, \quad N = \text{unit normal to } \partial \Omega
\]

but specified (non-zero) traction is more subtle; the statements

\[
P \cdot N = f \quad ("\text{dead load} \ f")
\]

\[
\tau \cdot \bar{n} = f \quad ("\text{live load} \ f")
\]

are different. (Dead loads are hand to apply in practice, since load must maintain its direction + magnitude limit set area regardless of deformation. Our scheme is use
long springs, e.g. for uniaxial tension

\[ \sum_{j=0}^{3} \frac{\partial}{\partial x_j} (T_{ij}) + f_i = 0 \quad (\text{work } f = \text{force per unit deformed vol}) \]

\[ \sum_{x=1}^{3} \frac{\partial}{\partial x_x} (P_{ix}) + f^R_i = 0 \quad \text{with } f^R = \frac{\partial}{\partial x_x} \text{det}(\frac{\partial x}{\partial x_x}) \quad \text{force/unit vol in rest coords} \]

Claim: \[ P_{ix} = \text{det} \left( \frac{\partial x}{\partial x_x} \right) \sum_{k} T_{ik} \frac{\partial x_k}{\partial x_x} \]

(In matrix notation: \[ P = \mathbf{I} \mathbf{I}^T (F^{-1})^T \] where \( \mathbf{F} = \frac{\partial x}{\partial x_x} \) and \( \mathbf{I} = \text{det } F \).)

\[ \text{Proof of Claim: } \sum_{i} \frac{\partial}{\partial x_i} (T_{ij}) + f_i = 0 \iff \int -\sum_{i} T_{ij} \frac{\partial q_i}{\partial x_j} + \sum f_i q_i \, dx = 0 \]

for all statically admissible \( \Phi \) (here \( i \) is fixed)
Change to \( X \) vars:
\[
\int \left[ - \sum_j T_{ij} \frac{\partial}{\partial X_i} \frac{\partial X_k}{\partial x_i} + f_{ij} \right] \left( \det \frac{\partial x_i}{\partial X_i} \right) \, dX = 0.
\]
\[
\sum_k \frac{\partial}{\partial X_k} P_{ik} + f_i = 0 \quad \text{with} \quad P_{ik} = \sum_j T_{ij} \frac{\partial X_k}{\partial x_i} \det \frac{\partial x_i}{\partial X_i}.
\]
[Warning: as noted earlier, \( P_{ik} \) is not symmetric.]

Another place where Eulerian vs. Lagr. is relevant in the comparison between elastodynamics + fluid dynamics.

Euler elastodynamics, in Lagr vars:
\[
\sum_k \frac{\partial}{\partial X_k} (P_{ik}) + m(X) g_i = m(X) \ddot{x}_i
\]
where
\[
m(X) = \text{mass per unit rest vol (density)}
\]
\[
m(X) g = \text{body force per unit rest vol}
\]
\[
\ddot{x} = \text{2nd deriv w/t, holding rest pos fixed}
\]

To recognize care of vars + momentum as consequences, we should write this in Eulerian coords. It's not to distinguish
\[ \frac{df}{Dt} = \text{change of } f \text{ wrt } t, \text{ holding } X \text{ fixed} \]
\[ \frac{\partial f}{\partial t} = \text{change of } f \text{ wrt } t, \text{ holding } x \text{ fixed} \]

By chain rule:
\[ \frac{df}{Dt} = \frac{\partial f}{\partial t} + \sum_i v_i \frac{\partial f}{\partial x_i} \quad \text{where } v_i = \frac{Dx_i}{Dt} \]

(we wrote \( x_i \) for \( v_i \) on page 4.5)

Claim: Equus of elasto-dynamics in Eulerian coodts are
\[ \frac{\partial f}{\partial t} + \sum_j \frac{\partial}{\partial x_j} (\rho v_j) = 0 \quad \text{[cons of mass]} \]
\[ \rho \left( \frac{\partial v_i}{\partial t} + v_j \frac{\partial v_i}{\partial x_j} \right) = \sum_j \frac{\partial}{\partial x_j} (\rho v_i) + \rho g_i \quad \text{[cons of momentum]} \]

where
\[ \rho (x,t) = \text{mass per unit detformed vol} \]
\[ v(x,t) = \text{velocity (at deformed position } x, \text{ time } t) \]

Ps 1st eqn:
\[ \rho \frac{df}{dt} \int_B m \, dX \quad \text{in any region } B \text{ of } \mathbb{R}^2 \]
\[ = \frac{\partial}{\partial t} \int_{x(B,t)} \rho \, dX \]
\[ = \int_{\partial x(B,t)} p \, v \cdot n \, d\mathbf{x} + \int_{x(B,t)} p_t \, d\mathbf{x} + \int_{x(B,t)} \text{div}_x (p\mathbf{v}) \, d\mathbf{x} \]

True for all \( B \) \( \Rightarrow \) \( p_t + \text{div}_x (p\mathbf{v}) = 0 \).

Note that this eqn is "purely kinematic" (i.e., it makes no use of eqns of motion, forces, etc.)

Pf of 2nd eqn: recall that \( \text{div}_x P = J \text{div}_x \mathbf{v} \),

with \( J = \det(\partial x/\partial \mathbf{x}) \). So

\[ m \ddot{\mathbf{x}} = \text{div}_x P + mg \Leftrightarrow J^{-1} m \ddot{\mathbf{x}} = \text{div}_x \mathbf{v} + J^{-1} mg \]

Now, \( J^{-1} = \mathbf{I} \) (by defn) \( \therefore \dot{\mathbf{x}} = \mathbf{v} = \frac{D\mathbf{v}}{Dt} = \mathbf{v}_t + \mathbf{v} \cdot \mathbf{v} \)

Subsit gives the asserted 2nd eqn (cons of mass).

\[ \underline{OK, let's turn to constitutive modeling} \]

Two viewpoints are possible:
(a) "Cauchy elasticity": specify Cauchy stress as the position + def gradient
\[ \tau = \nabla\cdot (X, F) \]

(b) "Hyperelasticity": specify \( P_{ij} = \frac{\partial W(X, F)}{\partial F_{ij}} \)
where
\[ W = "energy\; density" \]

View (b) is more restrictive — but therefore more useful (since it seems to be adequate). Often the body is homogeneous in its rest configuration; then constant reln w.r.t. vol of \( X \)

Always require structural const of "frame indifference."

a) for Cauchy elasticity: \( \nabla\cdot (RF) = R \nabla\cdot (F) R^T \)
for all con-preserving rotations \( R \)

b) for hyperelasticity: \( W(F) = W(RF) \) for all orientation-preserving rotations \( R \).

Interpret:
(a) observer in rotated coordinate system sees same basic stress-strain law
b) Rotations do no work

(We will ask you to show again if these relus, for hyperelasticity.)

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Many materials are isotropic. In hyperelasticity, isotropy \( \iff W(F) = W(FR) \) for any orientation-preserving map,

\[ \iff W(F) = \phi(\lambda_1, \lambda_2, \lambda_3) \quad \text{where} \quad \{\lambda_j\} \text{are the "principal stretches" (eigs of } (F^T F)^{1/2} \text{)}, \text{and } \phi \text{ is a symmetric function of its arguments.} \]

Corresp assertion for Cauchy elasticity:

\[ \text{isotropy } \iff \Sigma(FR) = \Sigma(F) \text{ for all } R. \]

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For strings it was natural to ask that \( v \rightarrow N(v) \) be monotone increasing. Similarly, for 3D elasticity it is natural to impose some structural conditions, e.g.

a) Equiv. elasticity states are elliptic
   (equil. of elastodynamics are hyperbolic)
or, moreglobally,

b) variationalprincipleofelastostaticsachieves itsminimum.

Discussion (b) would take us too far afield
(keyword: "W should be quasicovex."). Essence of (b) is that \( F \rightarrow W(F) \) should be (strictly)
"rank one convex"

\[
\sum \frac{\partial^2 W}{\partial F_i \partial F_j} \varepsilon_{ij} \varepsilon_{\beta} = C \varepsilon_{ij} \varepsilon_{\beta}^2 \]

for all \( \varepsilon_{ij}, \varepsilon_{\beta} \in \mathbb{R}^3 \). Intuition about why it matters: for a constant coefficient linear PDE system

\[
\sum \frac{\partial}{\partial x_\beta} \left( A_{i\alpha j} \frac{\partial x_\alpha}{\partial x_\beta} \right) = f_\alpha \quad \text{in } \mathbb{R}^n
\]

we can try to solve by Fourier transform

\[
- \sum A_{i\alpha j} \frac{\partial x_\alpha}{\partial x_\beta} \hat{x}_j (k) = \hat{f}_i (k)
\]

and (4) assures that the matrix \( \sum A_{i\alpha j} \frac{\partial x_\alpha}{\partial x_\beta} \)
(which has to be inverted) is positive definite.
Recall from above that in isotropic elasticity,

\[ W(F) = \Psi(\lambda_1, \lambda_2, \lambda_3) \]

where \( \lambda_i \) are roots of \((F^TF)^{1/2} + \Phi\) is symmetric for 3 vars.

In practice, a more useful rep (eg for calibration to experiments) is

\[
(*) \quad W(F) = \Psi(I, II, III)
\]

where \( I, II, III \) are the elementary symmetric functions of the roots of \( F^TF = C \), ie

\[
I = \text{tr} C = \lambda_1^2 + \lambda_2^2 + \lambda_3^2
\]

\[
II = \frac{1}{2} [(\text{tr} C)^2 - \text{tr}(C^2)] = \lambda_1^2 \lambda_2^2 + \lambda_1^2 \lambda_3^2 + \lambda_2^2 \lambda_3^2
\]

\[
III = \det C = \lambda_1^2 \lambda_2^2 \lambda_3^2
\]

Advantage of this rep: Piola-Kirchhoff stress involves a polynomial expression in elements of \( W \) and components of \( F \).

Analogous framework for constitutive modeling of Cauchy stress in an isotropic, homogeneous solid.
\[(\psi^*) \quad \mathbf{\tau}(F) = \Psi_0 \mathbf{I} + \Psi_1 \mathbf{B} + \Psi_2 \mathbf{B}^2 \]

where \( \mathbf{B} = FF^T \) and \( \Psi_i \) = suitable mix of \( \mathbf{I}, \mathbf{II}, \mathbf{III} \).

(Note: \( B = FF^T + C = F^TF \) have same eigenvalues, and therefore same invariants \( I, II, III \), though their eigenvectors are different.) HW 3 will ask you to show \((*) \Rightarrow (**)*\).

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Many materials (e.g., rubbers) are essentially incompressible. In an incompressible setting, role of elastostatics picks up an unknown function --- the pressure --- as a "Lagrange multiplier" for constraint of incompressibility. Constitutive law is then

\[\mathbf{\tau} = -p \mathbf{I} + \mathbf{\tau}^*(F)\]

where \( \mathbf{\tau}^* \) is given by a constitutive law (as above, but restricted to \( \det F = 1 \)) \& \( p \) is determined by balance of forces.

Explain this: starting from constrained variational

\[\min \quad 0 = \delta \left( \int W(\partial^2 \mathbf{u}/\partial x) \, dx \right) \quad \text{det}(\partial^2 \mathbf{u}/\partial x) = 1\]

\( = \text{by method of lag range multipliers, EL equ in}\)

formally
\[ 5 \int_{V} W(\frac{\partial Y}{\partial X}) + \psi(x) \left[ \det(\frac{\partial Y}{\partial X}) - 1 \right] \, dx = 0 \]

for some (unknown) \( \psi(x) \). Assume eqn in

\[ \sum_{\alpha} \frac{2}{\partial x_{\alpha}} \left( \frac{\partial W}{\partial F_{\alpha}} + \psi(x) \frac{\partial \text{det} F}{\partial F_{\alpha}} \right) = 0. \]

Our task is thus to show that the Cauchy stress \( \sigma \)

and the Piola-Kirchhoff stress \( \psi(x) \frac{\partial \text{det} F}{\partial F_{\alpha}} \)

have the form

\[ p(x) I, \]

In fact we'll show that \( p = -\psi(x) \).

Key is Cramer's rule, which says that

\[ \frac{\partial (\text{det} F)}{\partial F_{\alpha}} = \text{matrix of minors} = \tilde{J} (F^{-1})^T. \]

Recalling that \( P = \tilde{J} \tilde{C} (F^{-1})^T \) we get

\[ P = \psi \frac{\partial \text{det} F}{\partial F_{\alpha}} \Rightarrow \tilde{C} = \tilde{J}^{-1} \left[ \psi \tilde{J} (F^{-1})^T \right] F^T \]

\[ = \psi I. \]

as asserted.

Where to go from here?
1) Consider how inputs can be used to identify the constitutive law.

2) Discuss how nonlinearities of elasticity explain observable effects, e.g., relate thin radius + pressure when blowing up a balloon.

3) Discuss how linearization about \( x(0) = 0 \) leads to "linear elasticity.”

HW 3 will address (1) + (2), next lecture will turn to (3).