Mechanics - Lecture 2, 2/1/2012

Today's topic: a 1D bending theory (Euler's "elasticity"), both as 1st exposure to torque, and as an example of bifurcation.

Big picture about elastic beams + rods:

• In 2D, there is resistance to bending even for a sheet that's inextensible (eg. the "xerox paper problem," to be explained below and explored in HW 2)

• For a 2D sheet in \( \mathbb{R}^3 \), the situation is similar, except that the description of configurations + curvatures is more complicated (example: what determines the shape of a Hobies 6a/57)

• For a 1D (eg cylindrical rod) that lives in 3D, there can be curvature + also torsion (bending + also twisting)

To keep things simple, we'll focus on an inextensible beam (initially straight + uniform, eg a piece of paper or a ruler) deformed in the plane. (Antman's Chap 4 includes this + much more; Howell-Kozyreff-Ockendon Sect 4.9 has a concise treatment).

Note that essential kinematics of bending is
This: if a thin strip is wrapped to an annulus, with the midline wrapped isometrically, then lines parallel to the midline are stretched or shrunk due to effects of curvature.

OK, let's get started. Must discuss:

a) **kinematics** (description of deformation, e.g., curvature)

b) **statics** (forces + bending moments, and assoc balance laws)

c) **constitutive laws** (relation between curvature + moments, in this case)

**Kinematics**: For the special case of a 1D inextensible rod, this is easy: use osxL as the reference body, and \( \mathbf{r}(s) \in \mathbb{R}^3 \) as the deformed position. Then (by defn) inextensibility requires \( |\mathbf{r}_s| = 1 \), so

\[
\mathbf{r}_s = (\cos \theta(s), \sin \theta(s))
\]

and the rod's curvature is \( \theta'(s) \). (This is the analogue of strain in this setting. If the rod's "natural state" is straight then...
θ = 0 in the absence of loads or body couples.

Statics: slicing the rod at \( s = s_0 \), each side acts on the other by

i) a net force \( \bar{n}(s) \) (as with strip) but also

ii) an additional bending moment \( \bar{m}(s) \).

Definition of \( \bar{m} \): part of beam at \( s > s_0 \) exerts torque \( \bar{t}(s_0) \times \bar{n}(s_0) + \bar{m}(s_0) \) on the rest. (For deformation in x-y plane, \( \bar{m} = (0,0, M(s)) \).)

Suppose there's a body force (e.g., gravity) of \( \bar{f} \) per unit length. Then

\[
\text{balance of forces} \Rightarrow \bar{n}(s_1) - \bar{n}(s_0) + \int_{s_0}^{s_1} \bar{f}(s) \, ds = 0
\]

\[
\text{balance of torques} \Rightarrow \bar{m}(s_1) + \bar{t}(s_1) \times \bar{n}(s_1) - \bar{m}(s_0) - \bar{t}(s_0) \times \bar{n}(s_0)
+ \int_{s_0}^{s_1} \bar{t}(s) \times \bar{f}(s) \, ds = 0
\]

whence \( \bar{n}_d + \bar{f} = \overline{0} \)

\[
\bar{n}_d + (\bar{t} \times \bar{n})_d + \bar{t} \times \bar{f} = \overline{0}
\]

or equivalently

\[
\begin{align*}
\bar{n}_d + \bar{f} &= \overline{0} \\
\bar{m}_d + \bar{t} \times \bar{n} &= \overline{0}
\end{align*}
\]
Rules:
- Dynamic versions of these laws are cons of linear and angular momentum.
- Observe that if \( \ddot{\theta} = 0 \) then \( \bar{\tau} \) is constant.
- A 1D rod can be held in a bent position either by applying forces (corresponding in the picture) or by applying bending moments at each end.

Constitutive law: since the rod is inextensible, there is no constitutive law for \( \bar{\tau} \) (instead we get \( \bar{\tau} \) by integration of \( f \), with constants of integrals assumed to be zero).

The simplest law for \( \bar{m} \) is "physically linear":
\[
\bar{m} = (0, 0, M) \quad \Rightarrow \quad M = A \theta' \quad A = \text{constant}.
\]

We could of course take \( M \) to be a nonlinear function of \( \theta' \) instead. But the linear law is rich enough to be interesting (and it is reasonable, since if a rod is thin then substantial curvature still means relatively small change of length in the surfaces parallel to the median.)
When \( f = 0 \) this leads us to the "elasstica" (considered by Euler in 1727, but also J. Bernoulli in 1694; see Anthan for citations):

\[
\vec{r}_2 = (\cos \theta(s), \sin \theta(s))
\]

\[\vec{n} = 0 \Rightarrow \vec{n} = -\Lambda (\cos \alpha, \sin \alpha)\]

for some constants \( \Lambda, \alpha \)

\[\vec{m}_3 + \vec{r}_2 \times \vec{n} = 0, \quad \vec{m} = (0, 0, M) \Rightarrow\]

\[(0, 0, M_3) = \Lambda (\cos \theta, \sin \theta, 0) \times (\cos \alpha, \sin \alpha, 0)\]

\[
\Rightarrow M_3 = \Lambda \sin \theta \cos \alpha
\]

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\]

Constant law says \( M_3 = (A \theta'(s))' \). So,

ODE becomes (writing \( \phi = \theta(s) - \alpha \))

\[
[A \phi'(s)]' + \Lambda \sin \phi(s) = 0
\]

Note: in general \( \Lambda \) and \( \alpha \) are unknown, just like \( \phi(s) \); they must be determined from body data.

Example: deflection of a diving board (ignoring gravity). Let's take \( \phi = 0 \) to be clamped horizontally (\( \theta(0) = 0 \)) and suppose a downward
load \( F \) is applied at the R.H.S \( z = L \), which is otherwise free \((\theta'(L) = 0)\). Then \( \vec{n} = (0, -F) \) and 
\[
(0, 0, H_a) = (\cos \theta, \sin \theta, 0) \times (0, F, 0) = F \cos \theta
\]

so
\[
\theta_{\text{rad}} = F \cos \theta \quad \text{with} \quad \theta(0) = 0 \quad \theta_a(L) = 0
\]

A more or less exact solution is possible:
\[
\theta_{\text{rad}} \theta_a = F \theta_{\text{rad}} \cos \theta
\]

\[
\Rightarrow \quad (\theta_{\text{rad}})^2 = \frac{2F}{A} \sin \theta + \text{const}
\]

\[
= \frac{2F}{A} [\sin \theta + \sin \alpha], \quad \alpha = -\theta(L) > 0
\]

Since \( d\theta/da \) should be negative, a bit of arithmetic gives
\[
\int_0^{\theta(L)} \frac{d\theta}{\sqrt{\sin \alpha - \sin \theta}} = L \sqrt{\frac{2F}{A}}
\]

The value of \( \alpha \) is determined by solving (eg numerically) the eqn
\[
\int_0^{\theta(L)} \frac{d\theta}{(\sin \alpha - \sin \theta)^{1/2}} = L \sqrt{\frac{2F}{A}}
\]
The "xerox paper pbm" is only slightly different.

Question: describe the profile of a standard 8 1/2" x 11" sheet of paper, held at one edge so the tangent there is vertical.

Differences from the elastica:

* gravity matters
* must specify + use body coords

Now: \( \overrightarrow{\mathbf{n}} + \mathbf{f} = 0, \quad \mathbf{f} = f_0 (0, -1, 0) \)

and with these conventions

\[ s = \frac{L}{2}, \text{ is held} \]

The bc's are

* no force or moment at \( s = 0 \)
* specified angle \( \theta = -\pi/2 \) at \( s = L \)

Evidently: \( \overrightarrow{\mathbf{n}} = (a, b + f_0 s, 0) \) for some constants \( a, b \)

+ the bc at \( s = 0 \) \( \Rightarrow \) \( a = b = 0 \)
  \( \Rightarrow \overrightarrow{\mathbf{n}} = (0, f_0 s, 0) \)

Now eqn for \( \overrightarrow{\mathbf{n}} + \) linear const law +

remaining bc's =>

\[ A s^2 + f_0 s \cos \theta(s) = 0 \]

\( \theta(0) = 0, \quad \theta(L) = -\pi/2 \)
(On HW2 you'll be asked to estimate the magnitude of $\theta(x)$ in a piece of xerox paper.)

What can we do with this that is interesting? My choice: use it as an intuitive, physically natural example of bifurcation. (For a concise treatment like the one below, see Howells -- Kozyreff -- Ockendon 2 4.9.3; for a good intro to bifurcation in more generality see Ivar Stakgold's paper "Branching of solutions of nonlinear equations", SIAM Review 13 (1971) 289-332.)

Question 3 consider the elastica with compressive load $\lambda$. If $\lambda$ is large enough it will buckle. What is the critical load $\lambda_0$? How can we understand the buckled configurations?

Various choices of $bc$ are possible; let's choose

$\theta(0) = 0$ and LHS is clamped
($x_0$, $\tilde{\gamma}(0)$ is fixed, and $\tilde{\gamma}_a(0) = (1,0)$)

$\theta'(1) = 0$ RHS is "pinned" to loading device (applied load is $(2,0)$ but applied bending moment $= 0$)
Egn (derived earlier) is

\[
\frac{d}{ds}(A\theta_s) + \lambda \sin \theta(s) = 0
\]

Clearly \( \theta = 0 \) is a soln for any \( \lambda \). When \( \lambda \) is small we expect it to be stable; when \( \lambda \) is large enough we expect it to be unstable.

This pblm (with \( \lambda \) fixed) has a variational formulation: \( \theta(s) \) is a ctt pt of

\[
E = \int_0^1 \frac{1}{2} A \theta_s^2 + \lambda \cos \theta \, ds
\]

subject to \( \theta(0) = 0 \).

(The condition \( \theta'(1) = 0 \) arises as the "natural" bc at \( s = 1 \).) Interpret this as

\( E \) = elastic energy + work done by load since \( \int_0^1 \frac{1}{2} A \theta_s^2 \, ds \) = energy due to curvature, and \( \int_0^1 \lambda \cos \theta = \int_0^1 \lambda \frac{\theta_s}{\theta_s} \cdot (1, 0) = \lambda \left( \frac{\theta_1}{\theta_2} \cdot (1, 0) \right) \) is force \( \cdot \) (displacement of loaded pt).

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Natural physical expl is to increase \( \lambda \) gradually, starting from 0. Amounts mathematically.
to a "continuation method" by obtaining solutions \( \theta = \theta(x, x) \). Differentiating eqn \( \dot{\theta} \) with respect to \( x \) gives eqn \( \dot{\theta} = \partial \theta / \partial x \), which we can try to integrate (like an ode in \( x \)) to get \( \theta \). This general procedure is especially simple in this example: differentiating (*) with respect to \( x \) gives

\[
(*) \quad (A \dot{\theta}_+^T + \lambda \cos \theta) \dot{\theta} + \sin \theta = 0
\]

\[
\theta(0) = 0, \quad \dot{\theta}_+^T(1) = 0
\]

If \( \theta(0) = 0 \) then (*) implies \( \dot{\theta}(0) = 0 \) so long as \( \lambda < 1^{st} \) eigenvalue of linearized problem

\[
A \dot{\theta}_+^T + \lambda \dot{\theta} = 0, \quad \dot{\theta}(0) = \dot{\theta}_+^T(1) = 0.
\]

From now on let's take \( A = 1 \) for simplicity. Then 1st eqn is

\[
\lambda = \frac{\pi^2}{4}, \quad \text{corresp to eigenfunction}
\]

\[
\phi(x) = \sin \left( \frac{\pi}{2} x \right)
\]

Conclusions so far: when \( A = 1 \), crit load is \( \frac{\pi^2}{4} \). For general \( A > 0 \), crit load would be \( \frac{\pi^2}{4} A \) by same argument.

But \( \lambda \) doesn't stop at \( \lambda = \lambda_0 \) and neither should we. But we need a viewpoint that permits \( \theta = \theta(x, x) \) to be non-unique.
In fact, we'll show that the "bifurcation diagram" is (locally, near \( \lambda = \lambda_0 \)) like this:

\[
\begin{array}{c}
\text{stable} \\
\text{unstable}
\end{array}
\]

Variational perspective: for \( \lambda > \lambda_0 \) the variational problem has a saddle pt + 2 (nearby) local min.

\[
\lambda < \lambda_0 \quad \text{and} \quad \lambda > \lambda_0.
\]

Rigorous procedure is called "Lyapunov-Schmidt reduction" (I'll sketch it later). But basic idea can be captured by a more formal calculation:

\[
\lambda(\varepsilon) = \frac{\pi^2}{4} + \lambda_1 \varepsilon + \lambda_2 \varepsilon^2 + \cdots
\]

\[
\Theta(\varepsilon) = 0 + \varepsilon \Theta_1(\varepsilon) + \varepsilon^2 \Theta_2(\varepsilon) + \cdots
\]

and expand in powers of \( \varepsilon \). Full eqn is

\[
0 = (\varepsilon \Theta_1 + \varepsilon^2 \Theta_2 + \cdots) + \left( \frac{\pi^2}{4} + \lambda_1 \varepsilon + \lambda_2 \varepsilon^2 + \cdots \right) \sin(\varepsilon \Theta_1 + \varepsilon^2 \Theta_2 + \cdots)
\]

and we have (using \( \sin x \approx x - \frac{x^3}{6} + \cdots \))
\[
\sin (e^{\theta_1} + e^{2\theta_2} + \cdots) = e^{\theta_1} + e^{2\theta_2} + e^3 (\theta_3 - \frac{1}{6} \theta_3^3) + \cdots.
\]

So we get:

\[
\theta_1'' + \frac{\pi^2}{4}\theta_1 = 0 \quad \Rightarrow \quad \theta_1(0) = 0, \quad \theta_1'(0) = 0.
\]

\[
\Rightarrow \quad \theta_1(t) = f \Phi(t), \quad \Phi = \sin(\frac{\pi t}{2})
\]

\[f = \text{any constant}\]

at order \(e^2\):

\[
\theta_2'' + \frac{\pi^2}{4}\theta_2 = \xi_1 \theta_1 = -\xi_1 q \Phi(t), \quad \theta_2(0) = 0, \quad \theta_2'(0) = 0.
\]

\[\text{Soln exists iff } R+2 \text{ is a multiple of } L+H+2\]

ie if \(\int_0^1 \xi_1 q \Phi^2(t) \, dt = 0\). So (assuming \(q \neq 0\)) \(\xi_1 = 0\),

and \(\theta_2\) is again a multiple of \(\Phi\).

at order \(e^3\):

\[
\theta_3'' + \frac{\pi^2}{4}\theta_3 = -\xi_2 \theta_1 - \xi_2 (\theta_2 + \frac{\pi^2}{4} \theta_3^3), \quad \theta_3(0) = \theta_3'(0) = 0.
\]

\[\text{Soln exists iff}\]

\[
\int_0^1 -\xi_2 q \Phi^2 + \frac{\pi^2}{24} q^3 \Phi^4 = 0
\]

which simplifies to

\[
\xi_2 q = \frac{\pi^2}{52} q^3
\]

since \(\int_0^1 \sin^2(\frac{\pi t}{2}) = \frac{1}{2}, \quad \int_0^1 \sin^4(\frac{\pi t}{2}) = \frac{3}{8},\)

We could continue but there's no need: we've shown that the bit diagram is locally a parabola,
opening to the right since \( \frac{\pi^2}{22} > 0 \)

\[ \theta_0 = \pi/4 \]

\[ \lambda - \lambda_0 = \frac{\pi^2}{32} \theta^2 \]

\[ \lambda = (\theta - \theta_0) \]

Both is called "supercritical" because parabola opens to the right.

When imperfections are present, they break the symmetry of the bifurcation diagram, e.g.

Physical example: gravity or else a little intrinsic curvature will make the electric charge between the two (otherwise symmetry-related) ways to buckle. You'll be asked to do an example on HW 2.

Sketch how Liepman-Schmidt reduction makes our formal argument rigorous.
Let's look for

$$\theta = g \phi + \tilde{\theta} \quad \tilde{\theta} \perp \phi$$

where $\phi = 1^{st}$ eigenfunction $= \sin \left( \frac{\pi}{d} \right)$. Write eqn

$$\theta'' + \lambda \sin \theta = 0$$

as

$$\theta'' + \lambda_0 \theta + (\lambda - \lambda_0) \theta + \lambda (\sin \theta - \theta) = 0$$

i.e.

$$\theta'' + \lambda_0 \theta = - (\lambda - \lambda_0) \theta - \lambda (\sin \theta - \theta)$$

Consistency can then be

$$\lambda_0 \theta + \lambda (\sin \theta - \theta) \perp \phi$$

If this holds, there's a unique $\tilde{\theta} \perp \phi$ solving (1).

So we can view $\tilde{\theta} = \tilde{\theta} \perp \phi$ as being defined

for $\lambda$ near $\lambda_0$ and $\phi$ near $\phi^{1/2}$ by

$$\tilde{\theta}'' + \lambda_0 \tilde{\theta} = \phi \left[ - (\lambda - \lambda_0) \theta - \lambda (\sin \theta - \theta) \right]$$

$$\tilde{\theta} \perp \phi, \quad \tilde{\theta}(0) = \tilde{\theta}'(1) = 0$$

The eqn (1) gives the relation between $g + \lambda$

that describes the bifurcation. One can show

(using $\sin \theta - \theta \approx \frac{1}{6} \theta^3$) that

$$||\tilde{\theta}|| \leq C (g \phi^3)$$

The leading order character of bifurcation relation is

$$\int (\lambda - \lambda_0) (g \phi + \tilde{\theta}) \phi + \lambda_0 \left( \frac{-1}{6} g^3 \phi^3 \right) \phi = 0$$
\[ \lambda = \lambda_0 \int \phi^2 - \frac{1}{6} \lambda_0 \int \phi^4 = 0 \]

as obtained by formal expansion previously.

( Essence of this approach: in the nontrivial solutions, \( \theta = q \phi + \Phi \) is represented as a graph over the 1D axis \( q \phi \).)

See Stakgold's article for details and much amplification.